FAIR VALUATION OF ACTUARIAL LIABILITIES IN A BINOMIAL ENVIRONMENT

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Fair value of actuarial liabilities

1. INTRODUCTION: FAIR VALUES
2. STATE PRICES AND DEFLATORS
3. THE BINOMIAL CASE
4. APPLICATION TO FAIR VALUATION OF PENSION LIABILITIES
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1. Introduction: FAIR VALUES

✓International norms IAS / IFRS for all financial institutions in Europe

✓As from 01/ 01/2005

✓General principle of fair valuation of elements for assets as well as for liabilities

-IAS 19 : Employee benefits (pension plan)
-IAS 39 : Financial instruments
-IAS ?? : Insurance contracts

1. Introduction: FAIR VALUES

- Basic principle : from an historical or statutory accounting philosophy to fair value bases

- **Fair value** : price at which an instrument would be traded if a liquid market existed for this instrument

- ASSETS : market values

- LIABILITIES : ???

  If no market value : principle of estimation of future cash flows properly discounted and taking into account the different kinds of risk

1. Introduction: FAIR VALUES

- Need to develop good models of valuation especially for actuarial liabilities where there is a mixing between financial elements (optional elements, guarantee ,…) and insurance risk elements (mortality, disability,…)

- Consistency between modern financial pricing theory and classical actuarial models

- Even if for competition reasons methods of pricing could remain very classical , fair valuation will require new insights taking into account modern finance

2. STATE PRICES AND DEFLATORS

Single period model :

✓ one riskless asset :

$$S_0(1) = S_0(0)(1+r) \quad with \quad r = riskfree \quad rate$$

✓ d risky assets defined on a probability space :

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\} \quad with \quad p_j = P(\{\omega_j\}) \quad j = 1, \ldots, N$$

$$S_i(1) = (S_i(1,\omega_1), S_i(1,\omega_2), \ldots, S_i(1,\omega_N)) \quad i = 1, \ldots, d$$

✓classical assumption of absence of arbitrage opportunities

2. STATE PRICES AND DEFLATORS

✓STATE PRICE : random variable ψ such that for any asset :

$$S_i(0) = \sum_{j=1}^{N} \Psi_j \, S_i(1,\omega_j) \quad with \; \Psi_j = \Psi(\omega_j) \rangle 0$$

✓DEFLATOR : random variable D defined by :

$$D(\omega_j) = D_j = \frac{\Psi_j}{p_j}$$

2. STATE PRICES AND DEFLATORS

<u>Property 1</u> : for i=0 (riskfree asset):

$$i. \quad \sum_{j=1}^{N} \Psi_j = \frac{1}{1+r}$$

$$ii. \quad \sum_{j=1}^{N} p_j D_j = E(D) = \frac{1}{1+r}$$

2. STATE PRICES AND DEFLATORS

Property 2 : if X is a financial instrument on this market (replicable by the underlying assets) and giving for scenario j a cash flow X(1,j) then the initial value of this instrument is given by :

$$i.\quad X(0)=\sum_{j=1}^{N} X(1,j)\,\Psi_j$$

$$ii.\quad X(0)=\sum_{j=1}^{N} p_j D_j\, X(1,j)=E(D\,X(1))$$

Deflators are stochastic discounting factors depending on the financial scenario

2. STATE PRICES AND DEFLATORS

Property 3 : There exists a state price / a deflator if and only if there is no arbitrage opportunities; the deflator is unique if the market is complete

2. STATE PRICES AND DEFLATORS

Multiple periods model : discrete time model (t=0,1,…, T)

✓Riskfree asset:

$$S_0(t) = S_0(0)(1+r)^t \quad with \;\; r = riskfree \;\; rate$$

✓Risky assets :

$$S_i(t) = (S_i(t,\omega_1), S_i(t,\omega_2), \ldots, S_i(t,\omega_N)) \quad i = 1,\ldots,d$$

✓STATE PRICE :

$$S_i(0) = \sum_{j=1}^{N} \Psi_j(t) S_i(t,\omega_j) \quad with \; \Psi_j(t) = \Psi(\omega_j, t) \rangle 0$$

2. STATE PRICES AND DEFLATORS

✓ DEFLATOR :

$$D_j(t) = \frac{\Psi_j(t)}{p_j} = discount\ \ factor\ from\ \ t\ \ to\ 0\,if\ scenario\ j$$

✓Pricing : if X is a financial replicable instrument on this market generating successive stochastic cash flows :

$$\{C(t,\omega); t = 1,\ldots,T; \omega \in \Omega\}$$

Then the initial price of X can be written :

$$X(0) = \sum_{t=1}^{T} \sum_{j=1}^{N} C(t,\omega_j)\,\Psi_j(t)$$

2. STATE PRICES AND DEFLATORS

Or with deflators :

$$X(0)=\sum_{t=1}^{T}\sum_{j=1}^{N} p_j\, C(t,\omega_j)\, D_j(t)=\sum_{t=1}^{T} E(D(t)C(t))$$

CONCLUSION:

FAIR VALUE = Expected value of the discounted cash flows with respect to the historical probability measure (no risk neutral adjustment) but using stochastic discount factor instead of the riskfree rate.

3. THE BINOMIAL CASE

Single period model:

✓Risky asset :

$$S_1(1) = S_1(0) \cdot u \qquad \text{with probability p}$$

$$= S_1(0) \cdot d \qquad \text{with probability 1-p}$$

Absence of arbitrage opportunities if:

$$0 < d < 1 + r < u$$

Other form of the risky asset :

$$u = 1 + r + \lambda + \mu \qquad\qquad d = 1 + r + \lambda - \mu$$

3. THE BINOMIAL CASE

With condition : $0 < \lambda < \mu$

$\lambda = risk\ \ premium \qquad \mu = volatility$

<u>Equations of the STATE PRICE :</u>

For i=0: $(1+r)\Psi_1 + (1+r)\Psi_2 = 1$

For i=1: $u\Psi_1 + d\Psi_2 = 1$

3. THE BINOMIAL CASE

Solution for the STATE PRICE:

up $$\Psi_1 = \frac{1+r-d}{(1+r)(u-d)} = \frac{\mu-\lambda}{2\mu(1+r)}$$

down $$\Psi_2 = \frac{u-(1+r)}{(1+r)(u-d)} = \frac{\mu+\lambda}{2\mu(1+r)}$$

Safety principle :

$$\Psi_2 = \Psi_1 \quad if \quad \lambda = 0$$

$$\Psi_2 \rangle \Psi_1 \quad if \quad \lambda > 0 (normal\ case)$$

3. THE BINOMIAL CASE

Fair value in a binomial environment – single period :

If X is a financial instrument on this market with future stochastic cash flows given respectively by :

$$X(1,\omega_1) = X_1 \quad and \quad X(1,\omega_2) = X_2$$

Then the initial fair value of X is given by :

$$X(0) = X_1 \Psi_1 + X_2 \Psi_2$$

Or :

$$X(0) = \frac{1}{2}(X_1 + X_2)\frac{1}{(1+r)} + \frac{1}{2}(X_1 - X_2)\frac{\lambda}{\mu(1+r)}$$

3. THE BINOMIAL CASE

Fair value in a binomial environment – single period

Comparison with classical actuarial pricing :

$$fair\,value: X(0) = E(D\,X(1))$$

$$actuarial\,value: ??\ X(0) = \frac{1}{1+r} E(X(1))\ ??$$

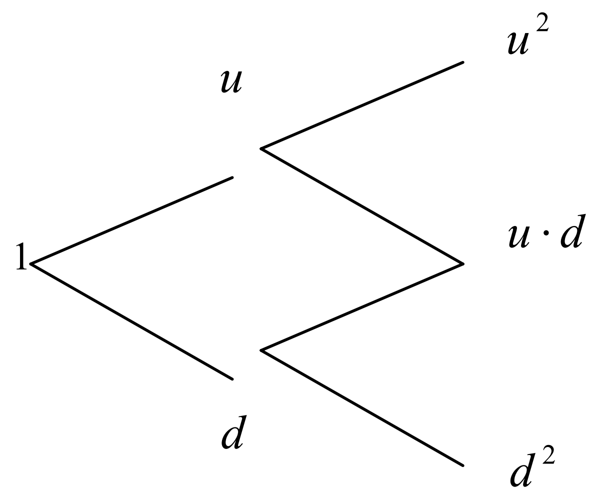
Application of the actuarial value for X= risky asset :

$$?\ S_1(0) = \frac{1}{1+r} E(S_1(1)) = \frac{1+r+\lambda}{1+r} S_1(0)$$

3. THE BINOMIAL CASE

Multiple periods model:

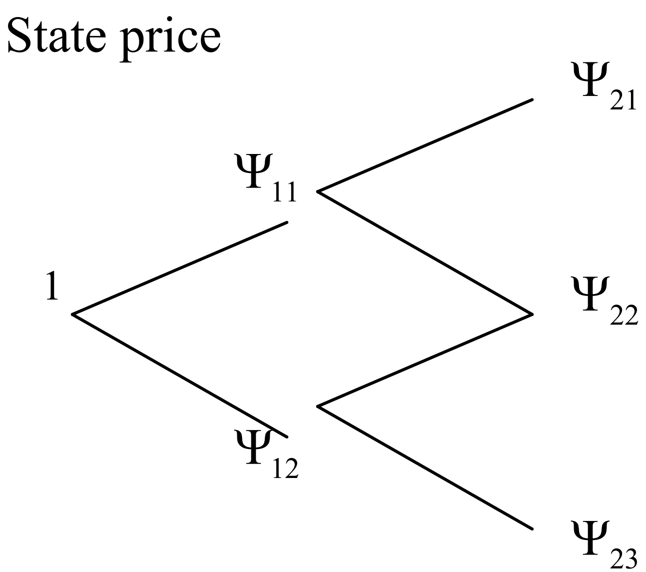
Risky asset



3. THE BINOMIAL CASE

Structure of STATE PRICES in multiple periods:

Assumption: financial product having successive cash flows depending only on the current situation of the market (no path dependant).



$$\Psi_{t\,j} = state\ price\ at\ time\ t \quad if\ scenario\ j$$

3. THE BINOMIAL CASE

Value of the STATE PRICES:

$$\Psi_{t\,j} = C_t^{j-1}\, \Psi_1^{\,j-1}\, \Psi_2^{\,t-j-1}$$

Where j-1 = number of up (j=1,..,t+1)
t-j-1 = number of down

And: C_t^{j-1} is the number of paths in the tree with j-1 up in t periods

3. THE BINOMIAL CASE

Fair valuation in multiple periods – binomial :

If X is a finacial instrument having successive cash flows in the tree given by :

$$X_{t\,j} = cash\ flow\ at\ time\ t\ if\ scenario\ j$$

Then the initial fair value of X is given by :

$$X(0) = \sum_{t=1}^{T} \sum_{j=1}^{t+1} X_{t\,j}\, \Psi_{t\,j}$$

4. FAIR VALUATION OF PENSION LIABILITIES

Joined work with Inmaculada DOMÍNGUEZ-FABIÁN (Universidad de Extremadura)

Purposes :

- How to valuate pension annuities not in terms of technical basis but in terms of market fair values;
- Influence of reversionary bonus on the level of provision;
- Sensitivity of the provision with respect to financial parameters;
- How to fix the technical interest rate.

4. FAIR VALUATION OF PENSION LIABILITIES

<u>Liability model</u> :

- Immediate lifetime annuity for an affiliate to a pension fund
- x : initial age at time t=0
- Liability to pay: 2 cases :

  1) *fixed annual pension :* L

  2) $L_{t,j} = amount\ to\ pay\ at\ time\ t\ for\ scenario\ j$

  (possibility to increase yearly the pension depending on the financial performances – asset side)

- Payment at the end of the year till death or during a fixed period of n years

4. FAIR VALUATION OF PENSION LIABILITIES

Liability model (2):
Actuarial first order bases :

$$i = technical\ discount\ rate$$

$$_{t}p_{x} = survival\ \ probability\ at\ time\ t$$

Technical provision for a constant pension (case 1):

$$L_{t,j} = L$$

$$_{n}V_{x} = L\, a_{x\,\overline{n}|} = L \sum_{t=1}^{n} {}_{t}p_{x} \frac{1}{(1+i)^{t}}$$

4. FAIR VALUATION OF PENSION LIABILITIES

<u>Asset model :</u>

Binomial model :

mixed financial strategy of the pension fund between riskless asset (r= riskfree rate) and risky asset (binomial model u / d)

γ : part invested in the risky asset

$1-\gamma$: part invested in the riskless asset

$(0 \leq \gamma \leq 1)$

4. FAIR VALUATION OF PENSION LIABILITIES

Back to the liability model (case 2):

Definition of the reversionary bonus in the case 2 of variable pensions

Used rule of bonus : comparison each year between the effective return of the assets and the riskfree rate; a part of this surplus is given back to the affiliate:

$0 \leq \beta \leq 1$: participation rate

4. FAIR VALUATION OF PENSION LIABILITIES

Yearly rate of increase of the pension (case 2) :

- ***If the risky asset is up :***

$$1+k=1+\beta\left(\frac{\gamma u+(1-\gamma)(1+r)}{(1+r)}-1\right)^{+}$$

or $$1+k=1+\beta\gamma\left(\frac{\lambda+\mu}{1+r}\right)$$

- ***If the risky asset is down*** :

$$1+l=1+\beta\left(\frac{\gamma d+(1-\gamma)(1+r)}{(1+r)}-1\right)^{+}=1$$

4. FAIR VALUATION OF PENSION LIABILITIES

Final form of the liabilities (case 2):

$$L_{t,j} = L \cdot (1+k)^{t-j+1}$$

Where t-j+1 is the number of times of up permitting to give a bonus.

As expected

THE LIABILITY DEPENDS ON TIME AND IS STOCHASTIC

4. FAIR VALUATION OF PENSION LIABILITIES

Computation of the fair value of the liabilities / case 1 :

(fixed pension)

$$FV(L\ )_{x,n} = \sum_{t=1}^{n} {}_t p_x \left[\sum_{j=1}^{t+1} L\ \Psi_{tj} \right]$$

$$= L \sum_{t=1}^{n} {}_t p_x \left[\sum_{j=1}^{t+1} \Psi_{tj} \right]$$

$$= L \sum_{t=1}^{n} {}_t p_x \left(\frac{1}{1+r}\right)^t \quad = L\, a^r{}_{x\,\overline{n|}}$$

4. FAIR VALUATION OF PENSION LIABILITIES

Computation of the fair value of the liabilities / case 2
(pension with reversionary bonus)

- Actuarial valuation : not so simple: liabilities not deterministic
- Fair valuation : general formula of valuation :

$$FV(L_k)_{x,n} = \sum_{t=1}^{n} {}_t p_x \left[\sum_{j=1}^{t+1} L_{t,j} \Psi_{tj} \right]$$

$$= L \sum_{t=1}^{n} {}_t p_x \left[\sum_{j=1}^{t+1} C_t^{j-1} \Psi_2^{j-1} \left(\Psi_1 (1+k) \right)^{t-j+1} \right]$$

$$= L \sum_{t=1}^{n} {}_t p_x \left[\left(\Psi_2 + \Psi_1 (1+k) \right)^t \right]$$

4. FAIR VALUATION OF PENSION LIABILITIES

Computation of the fair value of the liabilities / case 2 (pension with reversionary bonus)

$$FV(L_k)_{x,n} = L\sum_{t=1}^{n} {}_t p_x \left[\frac{1}{1+r}\right]^t \left[1+\beta\gamma\,\frac{\mu^2-\lambda^2}{2\mu(1+r)}\right]^t$$

$$= L\sum_{t=1}^{n} {}_t p_x \left(\frac{1}{1+i^*}\right)^t = L\, a^{i^*}_{x\,\overline{n|}}$$

4. FAIR VALUATION OF PENSION LIABILITIES

Equilibrium relation :

i* = equilibrium constant discount rate given by :

$$i^* = \left(r - \beta\gamma \frac{\mu^2 - \lambda^2}{2\mu(1+r)}\right) \Big/ \left(1 + \beta\gamma \frac{\mu^2 - \lambda^2}{2\mu(1+r)}\right)$$

$$\rightarrow if\ \ \beta = 0\ or\ \gamma = 0 : i^* = r$$

$$\rightarrow if\ \ \beta > 0\ and\ \gamma > 0 : i^* < r$$

5. NUMERICAL ILLUSTRATION

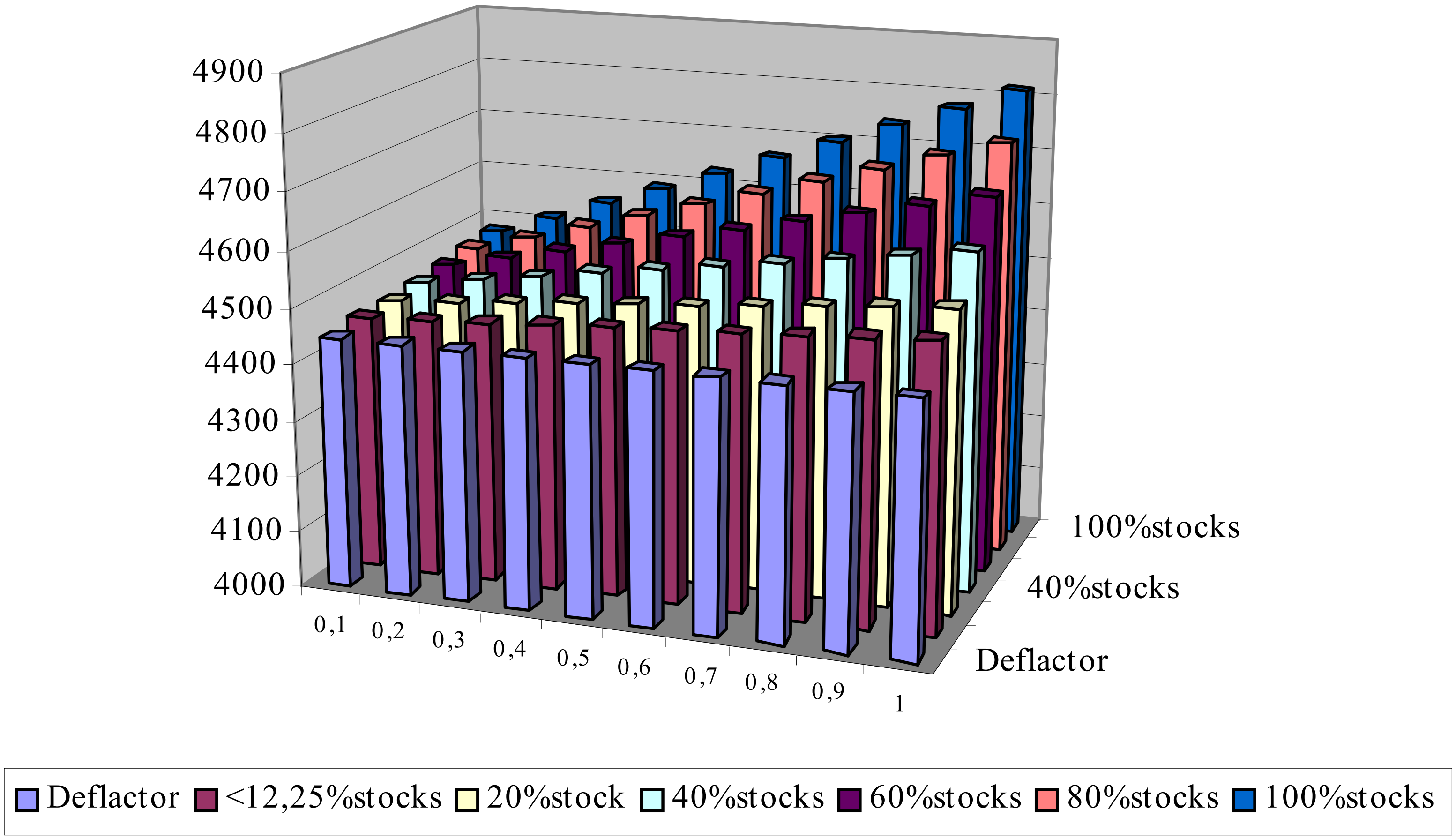
Numerical results :

Central scenario:
$u=1.1$ $d=0.99$
$r=0.03$

$i=0.025$
Mortality: GRM 95

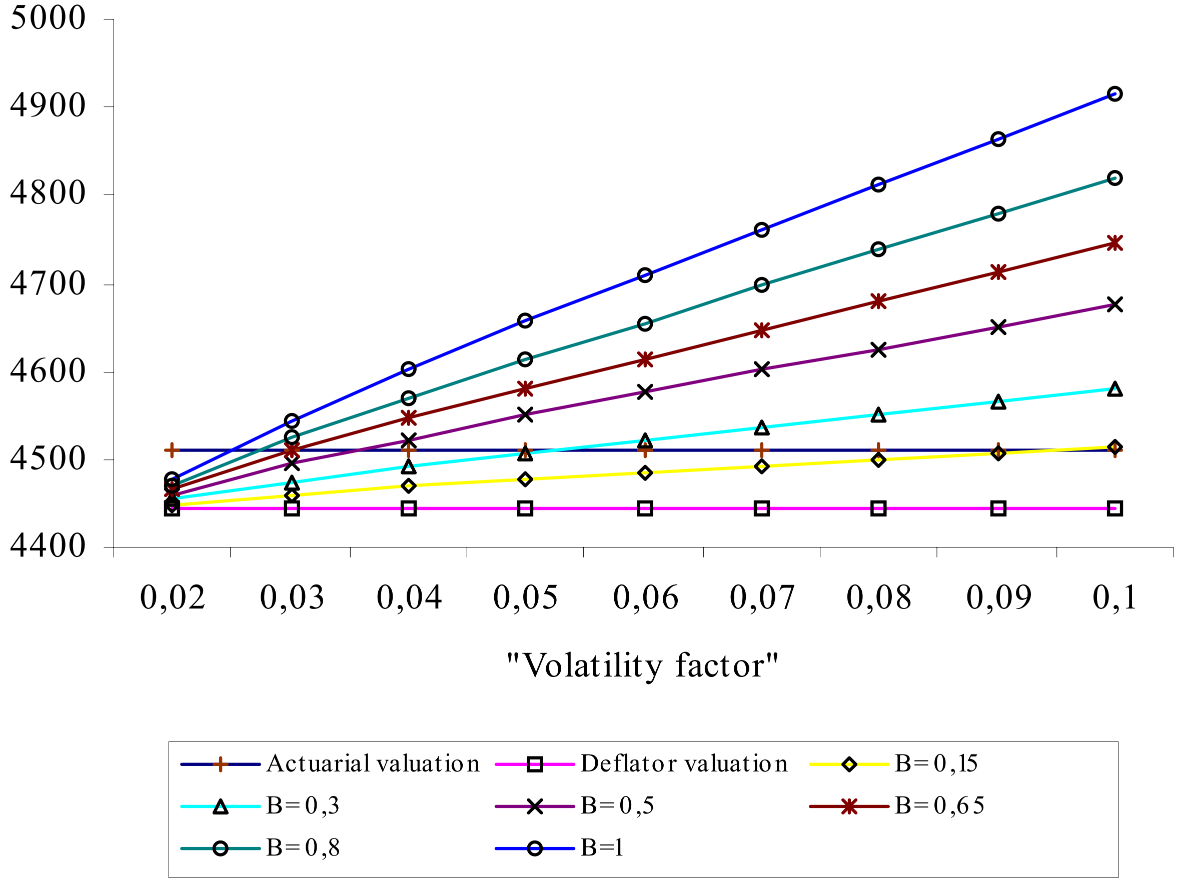
Risk premium : $\lambda = 0.02$
Volatility : $\mu = 0.06$

5. NUMERICAL ILLUSTRATION



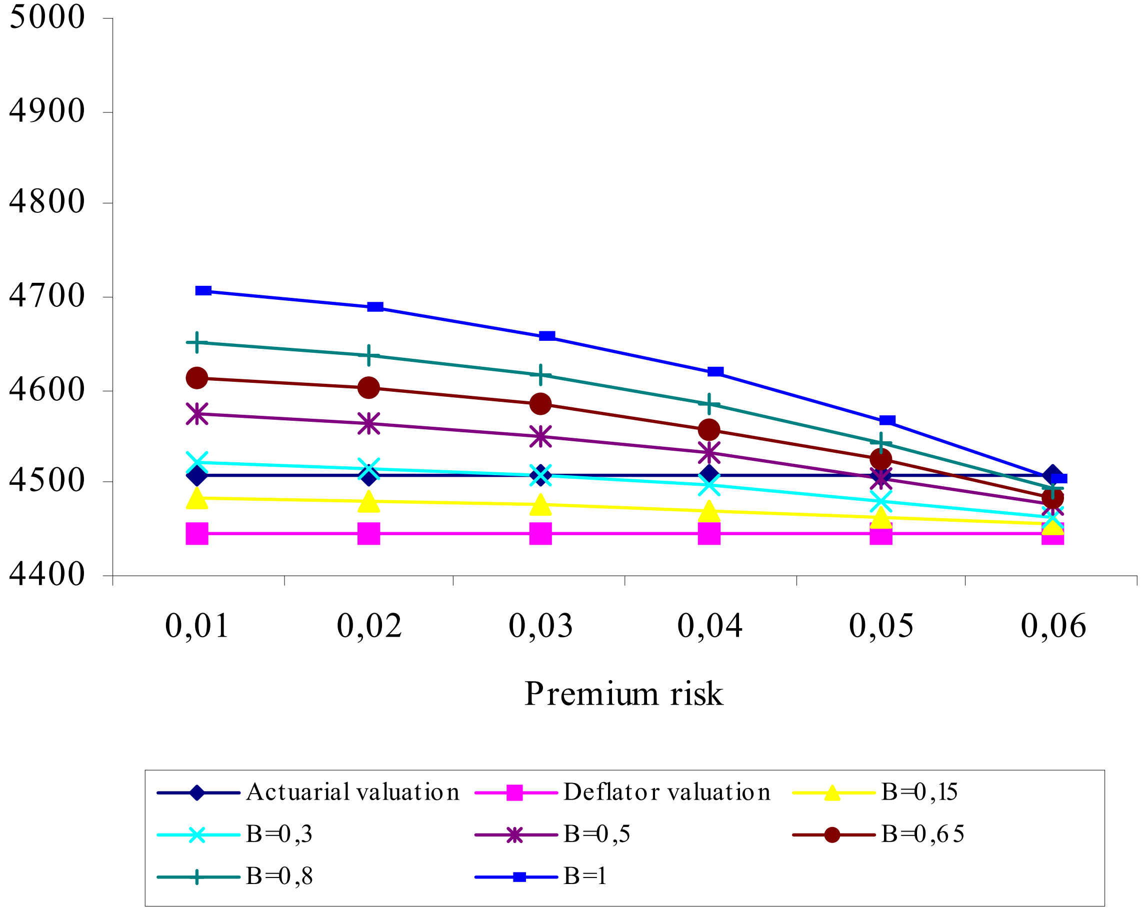
5. NUMERICAL ILLUSTRATION

Volatility sensitivity analysis : (60% in risky asset)



5. NUMERICAL ILLUSTRATION

Risk premium sensitivity analysis : (60% in risky asset)



5. NUMERICAL ILLUSTRATION

Value of the equilibrium discount rate : central scenario

1° for $\beta = 0.5$ and $\gamma = 0.6$:

i*= 2.21%

2° for $\beta = 1$ and $\gamma = 0.6$:

i*= 1.42%

3° for $\beta = 0.9$ and $\gamma = 0.4$:

i*= 2.05%

4° for $\beta = 1$ and $\gamma = 1$:

i*= 0.4%

$\lambda = 2\%$

$\mu = 6\%$

r=3%

i=2.5%

5. NUMERICAL ILLUSTRATION

Value of the equilibrium discount rate : other scenario
(less volatile)

1° for $\beta = 0.5$ and $\gamma = 0.6$:

i*= 2.60% versus 2.21%

2° for $\beta = 1$ and $\gamma = 0.6$:

i*= 2.21% versus 1.42%

3° for $\beta = 0.9$ and $\gamma = 0.4$:

i*= 2.52% versus 2.05%

4° for $\beta = 1$ and $\gamma = 1$:

i*= 1.68% versus 0.4%

$\lambda = 1\%$

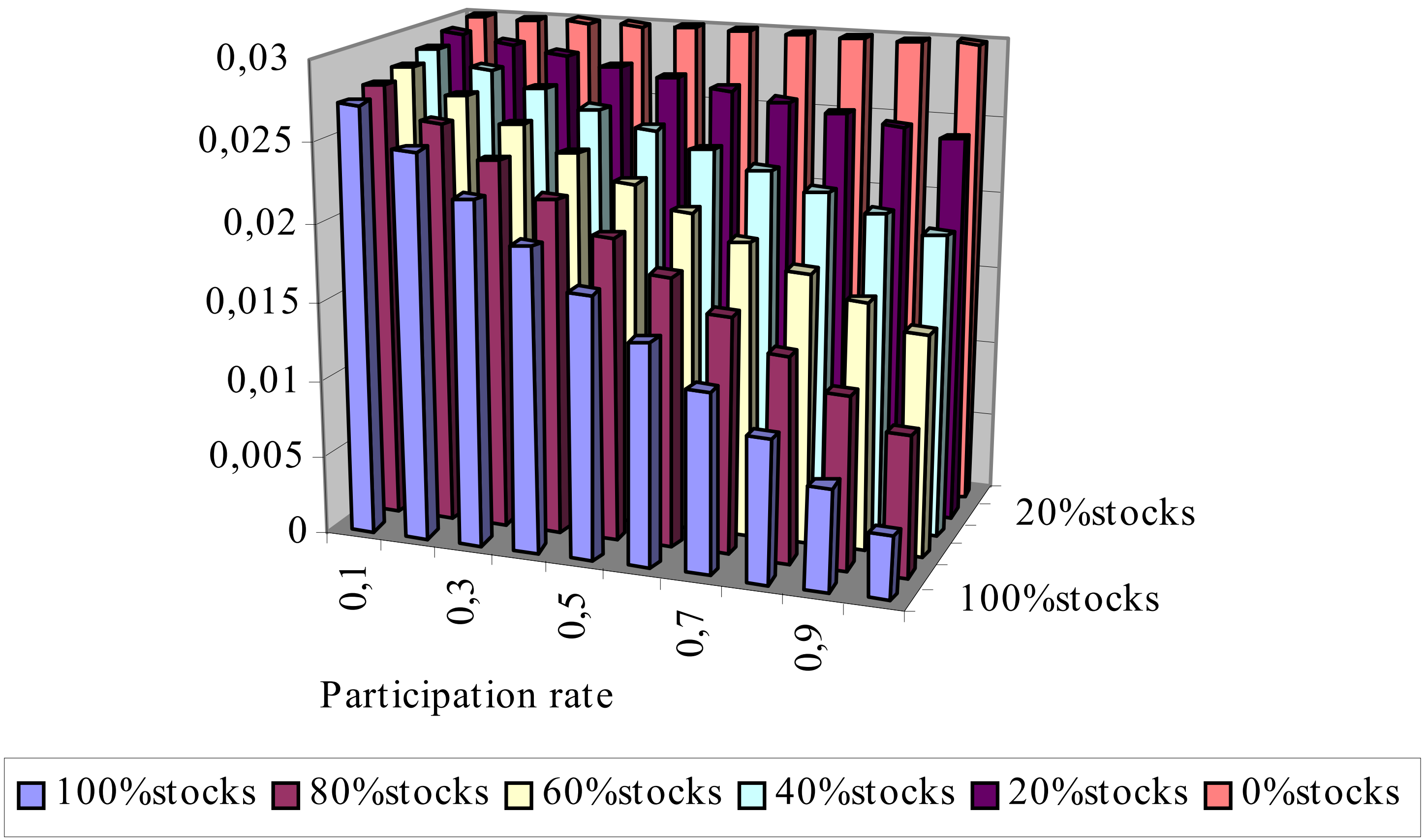
$\mu = 3\%$

r=3%

i=2.5%

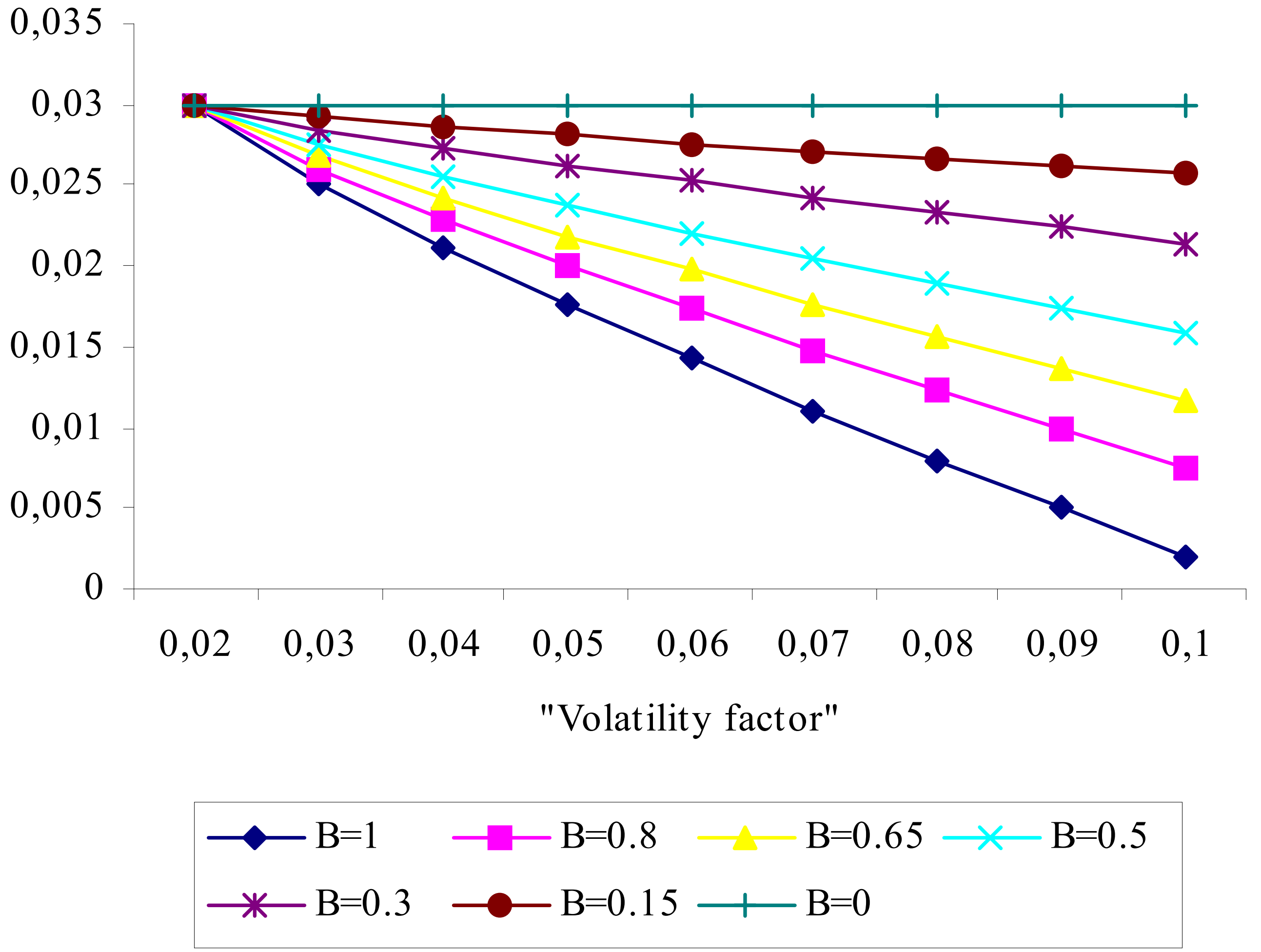
5. NUMERICAL ILLUSTRATION

EQUILIBRIUM TECHNICAL RATE(I)



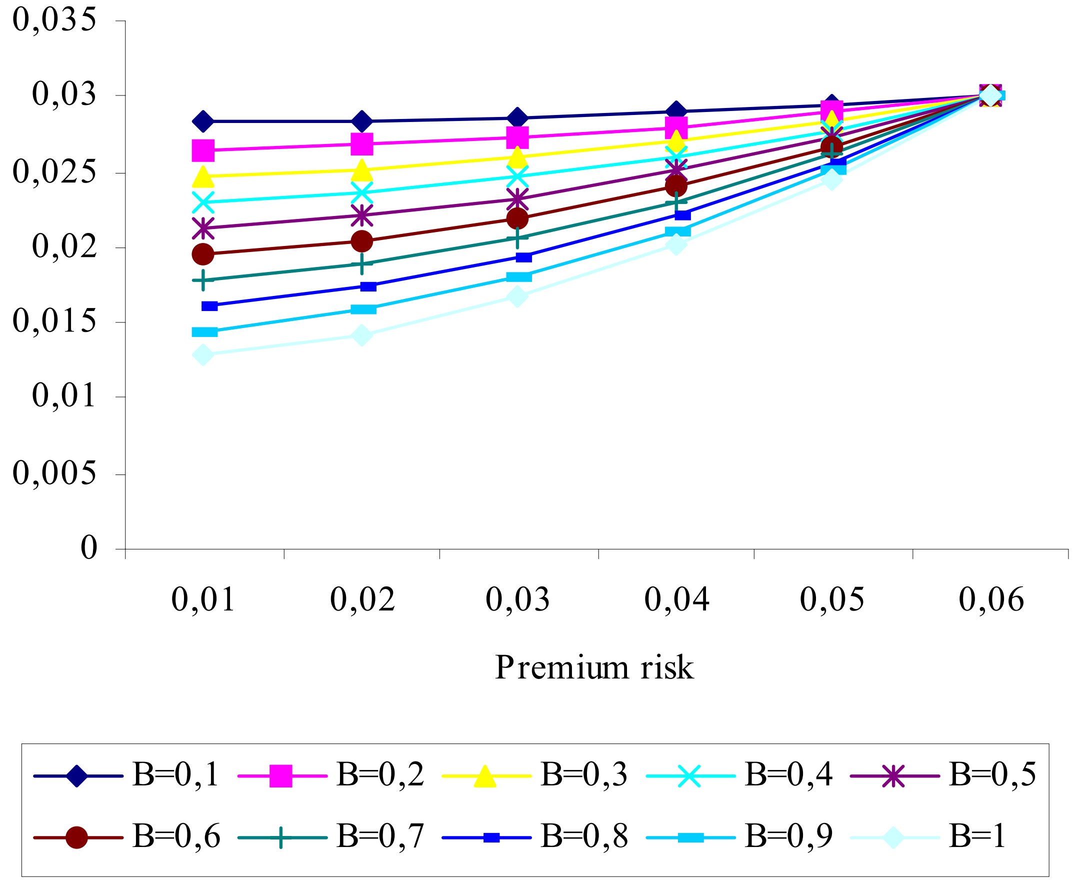
5. NUMERICAL ILLUSTRATION

EQUILIBRIUM TECHNICAL RATE(II)



5. NUMERICAL ILLUSTRATION

EQUILIBRIUM TECHNICAL RATE(III)



6.CONCLUSION

- State prices and deflators are an easy tool to valuate stochastic future cash flows correlated to future financial markets.

- The binomial model is a natural first step to introduce some uncertainty in the classical deterministic actuarial paradigm.

- One of the most important result is the way to valuate bonus and to define properly a technical interest rate taking into account at the same time the liability and the financial risk.

6.CONCLUSION

<u>Methodological extensions:</u>

More sophisticated asset models can be used in the same framework:

✓ Continuous time models for the risky asset

✓ Deterministic interest rate curve instead of a constant riskfree rate

✓ Uncertainty too on the interest rates (stochastic interest rate curve models)

6.CONCLUSION

Application extension :

▪A same methodology can be applied in order to valuate and price *life insurance contracts with profit.*

▪With the model it is possible **like here** :

- to compute fair value
- to find equilibrium values for the technical interest rate

but also :

- to price the surrender risk
- to consider lump sum contracts or contracts with periodical premiums and analyse guarantees on future payments.

6.CONCLUSION

…but this is perhaps for a next time…

THANKS