Australian Options

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Abstract

We study European options on the ratio of the stock price to its average and viceversa. Some of these options are traded in the Australian Stock Exchange since 1992, thus we call them Australian options. For geometric averages, we obtain closed-form expressions for option prices. For arithmetic means, we use different approximations that produce very similar results.

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Journal of Economic Literature classification: G13, C15

1 Introduction

Asian options are options on the average of asset prices. It is usually argued that they provide the following advantages: (a) they are cheaper than standard European options, as the average is less volatile than the asset price itself, (b) they prevent manipulation of the underlying asset price at the maturity date and (c) they are the adequate hedging instrument for traders who act continuously over finite periods.

Options on the ratio of the stock price to its average (or viceversa) are particular cases of Asian options. They have recently appeared as special types of variable purchase options (VPOs). VPOs were first issued in 1992 and have been traded since then on the Australian Stock Exchange. A VPO is an option that gives its holder the right to buy at maturity a stochastic number of shares that depends on the terminal stock price. This option can have more complex features like caps and floors on the number of shares.

Handley (2000) provides a detailed description of VPOs as well as pricing formulae, which are tested in Handley (2003). In the first article, the author describes Asian VPOs, in which the number of shares that can be bought at maturity depends on the average stock price. These options are shown to be equivalent to options on the ratio of the stock price to its average. Alternatively, we could define Asian VPOs in such a way that they are equivalent to options on the ratio of the average of the stock price to the stock price itself.

In this paper we price options on these ratios using both geometric and arithmetic (discrete- and continuous-time) means of stock prices that are assumed to follow a lognormal process. When the average is computed on geometric basis, these ratios are lognormally distributed at maturity, thus we obtain Black-Scholes-type formulae.

However, when the average is computed on arithmetic basis, the risk-neutral distribution of these ratios is, in general, unknown and we can not obtain closed-form expressions for the prices of these options.[1] This happens because the arithmetic average is the convolution of correlated lognormal random variables and its distribution is not known.[2]

This problem has been treated in the literature in different ways. Many studies use numerical techniques, such as the finite difference methods, as in Kemna and Vorst (1990) and Alziari *et al* (1997),[3] simulation, as in Kemna and Vorst (1990) or Vázquez-Abad and Dufresne (1998),[4] and the Fourier transform, as in Carverhill and Clewlow (1990) or Ju (1997).

A number of articles provide analytical solutions that approximate the price of these options. Examples of this literature include Yor (1992, 1993), Geman and Yor (1993), Curran (1994), De Schepper *et al* (1994), Eydeland and Geman (1995), Rogers and Shi (1995), Nielsen and Sandmann (1996, 1999, 2001), Fu *et al* (1999), Shirakawa (1999), and Dufresne (2000).

Jarrow and Rudd (1982) apply Edgeworth series expansion to option pricing when the risk-neutral distribution of the underlying asset at maturity is unknown. This method has been applied to Asian options by Turnbull and Wakeman (1991) and Ritchken *et al* (1993), among others. Some authors use only the first two moments in the Edgeworth series expansion, obtaining what is called the Wilkinson approximation. See, for example, Levy (1992) and Hansen and Jorgensen (2000).

Finally, Milevsky and Posner (1998) use the fact that the infinite sum of correlated lognormal random variables is reciprocal gamma distributed to obtain a closed-form solution for the value of arithmetic Asian options.[5] This formula is exact only when the average is computed continuously.

We price arithmetic Australian options using both the Wilkinson approximation and the gamma distribution. We also use Monte Carlo simulation with antithetic variables. The results show that option prices obtained with the three methods are quite similar. This is true even when the number of monitoring dates used to compute the average is small.

The rest of the paper is organized as follows. In Section 2 we generalize the Black-Scholes formula for option prices. Section 3 presents closed-form expressions for the prices of geometric Australian options and Section 4 presents approximations to the value of arithmetic ones. Section 5 summarizes and concludes. Technical details are relegated to the Appendix.

2 The Generalized Black-Scholes Model

Uncertainty is modelled by a filtered probability space $(\Omega, \mathbb{F}, P)$. The set of trading dates is $t \in [0, T]$. Let $Z = \{\ Z_t \mid t \in [0, T]\}$ be the price process for the underlying asset. The assumptions of the pricing model are as follows:

1. Markets are frictionless
   (a) No transaction costs when trading the stock or the option
   (b) No taxes
   (c) No penalties to short selling
   (d) All assets are perfectly divisible
2. Security trading is in continuous time

3. The term structure of interest rates is flat and known with certainty. Let $\beta = \{\ \beta_t;\ t \in [0,T]\ \}$ be the value process for a banking account defined by

$$d\beta_t = r\beta_t dt, \tag{1}$$

where $t \in [0,T],\ \beta_0 = 1$ and the risk-free interest rate, r, is constant

4. The asset offers a continuous dividend yield of δ in the interval $[0,T]$

5. The asset price follows a GBM process

$$dZ_t = \mu_Z Z_t dt + \sigma_Z Z_t d\widetilde{W}_t, \tag{2}$$

where $(Z_t, t) \in (0,\infty) \times [0,T]$, μ_Z and σ_Z are constants and $\widetilde{W}_t$ is a standard Wiener process. Usually, σ_Z^2 is referred to as the logarithmic variance parameter of the asset.

Under the risk-neutral probability measure, the process (2) becomes

$$dZ_t = \alpha_Z Z_t dt + \sigma_Z Z_t dW_t$$

where α_Z is the (constant) risk-neutral drift of the process.

The solution for this process is given by

$$Z_t = Z_0 \exp\left\{\left(\alpha_Z - \frac{1}{2}\sigma_Z^2\right) t + \sigma_Z W_t\right\} \tag{3}$$

Therefore, Z_t follows a lognormal process. Moreover, it is straightforward to show that

$$[\ln Z_u] \mid Z_t \sim N\left(\ln(Z_t) + \left(\alpha_Z - \frac{1}{2}\sigma_Z^2\right)(u-t), \sigma_Z^2(u-t)\right), \quad u > t \tag{4}$$

Lemma 1 *The moments of the variable Z_t under the risk-neutral measure are the following:*

$$\begin{aligned} E(Z_t) &= Z_0 e^{\alpha_Z t} \\ V(Z_t) &= [E(Z_t)]^2 \left[e^{\sigma_Z^2 t} - 1\right] \\ Cov(Z_t, Z_s) &= Z_0^2 e^{\alpha_Z (t+s)} \left[e^{\sigma_Z^2 s} - 1\right],\ s < t \end{aligned}$$

Proof: It follows from (3) and the properties of the lognormal distribution (see Lemma 5 in the Appendix for more details). □

The following proposition generalizes the Black-Scholes option pricing formula:

Proposition 1 *The price at time 0 of an European call option on Z that matures at time T and with strike price K is given by*

$$C(Z, 0, T, K) = e^{-rT} E(Z_T) N(d_1) - K e^{-rT} N(d_2) \tag{5}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(e^{-\alpha_Z T} E(Z_T)/K\right) + \left(\alpha_Z + \frac{1}{2}\sigma_Z^2\right) T}{\sigma_Z \sqrt{T}} \\ d_2 &= d_1 - \sigma_Z \sqrt{T} \end{aligned}$$

Proof: See the Appendix. □

The prices of European put options can be easily obtained using the put-call parity:

$$P(Z, 0, T, K) = C(Z, 0, T, K) - e^{-rT} E(Z_T) + K e^{-rT} \tag{6}$$

Let S_t denote the stock price at time t. We suppose that S_t follows the risk-neutral process

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \tag{7}$$

where q is the continuous dividend yield of the stock and σ is a constant.

The solution for this process is given by

$$S_t = S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\} \tag{8}$$

Note that application of Proposition 1 to the stock price process (7) leads to the Black-Scholes formula adjusted by dividends, as derived by Merton (1973).

3 Geometric Australian Options

We consider n monitoring dates so that the time interval $[0, T]$ is partitioned in the following way:

$$\{t_0 = 0 < t_1 < t_2 < \cdots < t_n = T\},\ t_i - t_{i-1} = \frac{T}{n} = \Delta t,\ \forall\, i = 1, \cdots, n.$$

Let $S = \{\ S_{t_i} \equiv S_i,\ i = 0, 1, \cdots, n\ \}$ be the price process for the stock. We define the geometric mean of the n stock prices $S_1, \cdots, S_n$ as

$$G_n == \left(\prod_{i=1}^{n} S_i\right)^{\frac{1}{n}}, \quad G_0 \equiv S_0$$

Using (8), we have

$$G_n = S_0 \exp\left\{\left(r - q - \frac{1}{2}\sigma^2\right) \frac{n+1}{2}\Delta t + \frac{\sigma}{n}\sum_{i=1}^{n} W_{t_i}\right\} \tag{9}$$

Looking at (8) and (9), and using $t_n = n\Delta t$, we have

$$\frac{S_n}{G_n} = \exp\left\{\left(r-q-\frac{1}{2}\sigma^2\right)\frac{n-1}{2}\Delta t + \frac{\sigma}{n}\left[n\,W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right]\right\} \quad (10)$$

$$\frac{G_n}{S_n} = \exp\left\{-\left(r-q-\frac{1}{2}\sigma^2\right)\frac{n-1}{2}\Delta t - \frac{\sigma}{n}\left[n\,W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right]\right\} \quad (11)$$

It is clear from (9)-(11) that the geometric average and both ratios are lognormally distributed. Thus we can apply Proposition 1 to obtain the next result.

Proposition 2 *We consider European call options on S_n/G_n and G_n/S_n that mature at time T and with strike price K. The prices at time 0 of these options are given by expression (5), where the expected value and the logarithmic variance of the asset at maturity are given by the following table:*[6]

Z_n	$E(Z_n)$	$\sigma_Z^2 T$
G_n	$S_0 \exp\left\{\left(r-q-\frac{n-1}{6n}\sigma^2\right)\frac{n+1}{2n}T\right\}$	$\frac{(n+1)\left(n+\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$
S_n/G_n	$\exp\left\{\left(r-q-\frac{n+1}{6n}\sigma^2\right)\frac{n-1}{2n}T\right\}$	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$
G_n/S_n	$\exp\left\{-\left(r-q-\frac{5n-1}{6n}\sigma^2\right)\frac{n-1}{2n}T\right\}$	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\,\sigma^2 T$

Proof: These moments are obtained using (9)-(11), the properties of the lognormal distribution, and Lemma 6 in the Appendix. □

The expression obtained in this Proposition is the Black-Scholes formula with a volatility parameter σ_Z and a continuous dividend yield $\delta = r - \alpha_{Z_n}$ as given by the next table:

Z_n	δ
G_n	$\left(\frac{n-1}{n+1}r + q + \frac{n-1}{6n}\sigma^2\right)\frac{n+1}{2n}$
S_n/G_n	$\left(\frac{n+1}{n-1}r + q + \frac{n+1}{6n}\sigma^2\right)\frac{n-1}{2n}$
G_n/S_n	$\left(\frac{3n-1}{n-1}r - q - \frac{5n-1}{6n}\sigma^2\right)\frac{n-1}{2n}$

Note that the prices of these options do not depend on the current stock price, S_0.

Obviously, when $Z_n = S_n$, we obtain the Black-Scholes formula which does not depend on the partition of the time interval. When $Z_n = G_n$, we get the price derived by Turnbull and Wakeman (1991) and Ritchken *et al* (1993).

The effect of the number of monitoring dates (n) on the expected value of the asset is not clear, since it depends on the relationship among r, q and σ. However, the effect of n on the logarithmic variance, $\sigma_Z^2 T$, is quite obvious as can be seen in Figure 1. The parameter values are $\sigma = 0.2$ and $T = 1$.

[Insert Figure 1 about here]

We observe the following facts:

- The logarithmic variance of the stock price is constant.
- The logarithmic variance of G_n is equal to that of the stock price when $n = 1$. Then it decreases with n, and converges to $\frac{\sigma^2}{3}T$, the logarithmic variance of the continuous geometric average, obtained by Kemna and Vorst (1990).
- The logarithmic variances of S_n/G_n and G_n/S_n are equal. This is due to the relationship between the variances of a lognormal variable and its reciprocal (see expressions (38)-(40)). This variance increases with n and converges to $\frac{\sigma^2}{3}T$.

It can also be of interest to study the effect of σ and T on option prices. We leave this analysis for the case of continuous-time means.

Tables 1 and 2 show call and put option prices respectively (multiplied by 100) for different cases and monitoring dates. Call prices are computed using Proposition 2. Put prices are obtained applying expression (6). The interest rate is 10% and the stock dividend yield is 3%. We include the stock price (S_n) and its geometric average (G_n) as underlying assets as a reference. In both cases, we assume that the initial stock price (S_0) is 1.

[Insert Tables 1 and 2 about here]

We see that the value of Australian options are relatively similar to those of geometric Asian options.

For one monitoring date, options on G_n have the same value as those on the stock. Moreover, Australian options are equal to options on the unity, and their values are given by $\exp\{-rT\}\max\{1-K,0\}$.

Interestingly, option prices do not necessarily increase with the volatility of the stock price (σ) either. This is also true for standard geometric Asian options. For example, from Table 1 we have that when $T = 0.5, K = 0.8$, and $n = 1,000$ the call option on G_n has a value of 20.548 and 20.538 for $\sigma = 0.2$ and 0.4, respectively.[7]

We see that option prices do not necessarily increase with time to maturity (T). For example, when $\sigma = 0.2, K = 0.8$, and $n = 100$, Table 1 shows that the call option on G_n/S_n has a value of 18.161 and 16.580 for maturities of 0.5 and 1.0 years, respectively.

The tables show that option prices do not change monotonically with n. For instance, in Table 1 we have that when $\sigma = 0.2, T = 0.5$, and $K = 1.1$, call options prices on G_n are 3.175, 0.905, and 0.737, when $n = 1, 10$ and 100, respectively.

The effect of a change in the exercise price (K) is as expected: call prices decrease and put prices increase with K.

As additional reference, the Black-Scholes call option prices (dividend yield = 0) in the four cases studied in Table 1 are 24.027, 27.993, 26.081, and 3.743, and the Black-Scholes put option prices corresponding to Table 2 are 3.400, 3.753, 8.703, and 8.378, respectively.

We now define the continuous geometric average of the stock price over the interval $[0, T]$ as

$$G_T = \exp\left\{\frac{1}{T}\int_0^T \ln(S_t)\, dt\right\}$$

Using (8), we have

$$G_T = S_0 \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\int_0^T W_t dt\right\} \quad (12)$$

Looking at (8) and (12), we have

$$\frac{S_T}{G_T} = \exp\left\{\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T + \frac{\sigma}{T}\left[T\, W_T - \int_0^T W_t dt\right]\right\} \quad (13)$$

$$\frac{G_T}{S_T} = \exp\left\{-\frac{1}{2}\left(r - q - \frac{1}{2}\sigma^2\right)T - \frac{\sigma}{T}\left[T\, W_T - \int_0^T W_t dt\right]\right\} \quad (14)$$

From (12)-(14) we see that the geometric average and both ratios are lognormally distributed. Thus, to price options on these assets, we just need the moments of their risk-neutral distributions.

Proposition 3 *We consider European call options on S_T/G_T and G_T/S_T that mature at time T and with strike price K. The prices at time 0 of these options are given by expression (5), where the expected value and the logarithmic variance of the asset at maturity are given by the following table:*[8]

Z_T	$E(Z_T)$	$\sigma_Z^2 T$
S_T	$S_0 \exp\{(r-q)T\}$	$\sigma^2 T$
G_T	$S_0 \exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$
S_T/G_T	$\exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$
G_T/S_T	$\exp\left\{-\frac{1}{2}\left(r-q-\frac{5}{6}\sigma^2\right)T\right\}$	$\frac{\sigma^2}{3}T$

Proof: These moments are obtained using (12)-(14), the properties of the lognormal distribution, and Lemma 7 in the Appendix. □

The option pricing formula given in this Proposition corresponds to a continuous dividend yield $\delta = r - \alpha_{Z_T}$ as given by the next table:

Z_T	δ
S_T	q
$G_T,\ S_T/G_T$	$\frac{1}{2}\left(r+q+\frac{1}{6}\sigma^2\right)$
G_T/S_T	$\frac{1}{2}\left(3r-q-\frac{5}{6}\sigma^2\right)$

Notice that:

- The logarithmic variances of both ratios are equal to the one derived by Kemna and Vorst (1990) for the continuous geometric average. The intuition for this result is that, with infinite monitoring dates, the volatility of the ratio depends only on the volatility of the average. This value increases with σ and is one third of the variance in the Black-Scholes formula.

- The expected value of G_T is S_0 times the expected value of S_T/G_T.

- The expected values of S_T/G_T and G_T/S_T do not depend on the current stock price, S_0.

- The expected values of G_T and S_T/G_T are smaller than that of S_T.

Figure 2 shows $E(Z_T)$ as a function of σ. The parameter values are $r = 0.1, q = 0.03$, and $T = 1$. We assume $S_0 = 1.2$.

[Insert Figure 2 about here]

We observe that the expected values of G_T and S_T/G_T decrease with σ, while that of G_T/S_T increases with σ.[9]

Since the logarithmic variance of the assets studied increase with σ and T, and their expected values also depend on these values, we have that option prices can decrease with volatility or time to maturity. This surprising result is described in more detail in the next lemma that shows the theta and vega for these options.

Lemma 2

1. *The theta of a call option on Z_T is given by*

$$\theta_C(Z_T) = e^{-rT}\left[(\alpha_{Z_T} - r)E(Z_T)N(d_1) + KrN(d_2) + \frac{\sigma_Z}{2\sqrt{T}}E(Z_T)N'(d_1)\right] \tag{15}$$

2. *The theta of a put option on Z_T is given by*

$$\theta_P(Z_T) = e^{-rT}\left[(r - \alpha_{Z_T})E(Z_T)N(-d_1) - KrN(-d_2) + \frac{\sigma_Z}{2\sqrt{T}}E(Z_T)N'(d_1)\right] \tag{16}$$

3. *The vega of a call option on Z_T is given by*

$$\nu_C(Z_T) = e^{-rT}E(Z_T)\sqrt{T}\left[\frac{\partial \alpha_Z}{\partial \sigma}\sqrt{T}N(d_1) + \frac{\partial \sigma_Z}{\partial \sigma}N'(d_1)\right] \tag{17}$$

4. The vega of a put option on Z_T is given by

$$\nu_P(Z_T) = e^{-rT} E(Z_T)\sqrt{T}\left[\frac{\partial \sigma_Z}{\partial \sigma} N'(d_1) - \frac{\partial \alpha_Z}{\partial \sigma}\sqrt{T} N(-d_1)\right] \qquad (18)$$

Proof: The results for call options are obtained differentiating with respect to T or σ in Proposition 1. For put options, the generalized put-call parity (6) is used. □

From (15), we see that if $q > 0$, $\alpha_{Z_T} - r < 0$ for all the assets except for G_T/S_T. Consequently, the sign of theta for call options on these assets is undetermined. If $q = 0$, we have that $\alpha_{Z_T} - r = 0$, and the stock call price increases with T. For G_T/S_T, we have that $\alpha_{Z_T} - r > 0 \Leftrightarrow r < \frac{1}{3}\left(q + \frac{5}{6}\sigma^2\right)$. In this case, an increase in T leads to a higher call option price.

For put options, theta can be positive or negative, depending on parameter values (see (16)). For all the assets except G_T/S_T, we have $r - \alpha_{Z_T} \geq 0$. Thus, if the exercise price is low enough, the put price on these assets will increase with T, while for large strikes the opposite will take place. For G_T/S_T, we have that $r - \alpha_{Z_T} > 0 \Leftrightarrow r > \frac{1}{3}\left(q + \frac{5}{6}\sigma^2\right)$. In this case, an increase in T can lead to a higher put price if the exercise price is small.

Figure 3 plots geometric Australian option prices as a function of time to maturity. The averages are computed with infinite monitoring dates. The exercise price is $K = 0.8$ for calls and $K = 1.2$ for puts. The other parameters are $r = 0.1, q = 0.03, \sigma = 0.2$.

[Insert Figure 3 about here]

We see that, in this case, the price of a call option on S_T/G_T increases with T. However, the price of a call option on G_T/S_T decreases with T. The latter result is due to the fact that $r > \frac{1}{3}\left(q + \frac{5}{6}\sigma^2\right)$, so that $\alpha_{Z_T} - r < 0$ and $\partial C(.)/\partial T$ can be negative.

Since the exercise price for the put options is relatively high ($K = 1.2$), we see that the price of the put option on S_T/G_T decreases with T. The same occurs for a put option on G_T/S_T when time to maturity is small (between 0 and 0.65 years). For higher T, the put price increases. When $T > 1.5$, the put price decreases again. Interestingly, if we reduce the exercise price to $K = 1.1$, the put price increases for all T.

Using Proposition 3 and Lemma 2, we obtain the following result.

Corollary 1

1. *The vega of a call option on the assets studied is given by*

$$\begin{aligned}
\nu_C(S_T) &= S_0\sqrt{T}N'(d_1) > 0 \\
\nu_C(G_T) &= e^{-rT}E(G_T)\sqrt{T}\left[-\frac{1}{6}\sigma\sqrt{T}N(d_1) + \frac{1}{\sqrt{3}}N'(d_1)\right] \\
\nu_C\left(\frac{S_T}{G_T}\right) &= e^{-rT}E\left(\frac{S_T}{G_T}\right)\sqrt{T}\left[-\frac{1}{6}\sigma\sqrt{T}N(d_1) + \frac{1}{\sqrt{3}}N'(d_1)\right] \\
\nu_C\left(\frac{G_T}{S_T}\right) &= e^{-rT}E\left(\frac{G_T}{S_T}\right)\sqrt{T}\left[\frac{5}{6}\sigma\sqrt{T}N(d_1) + \frac{1}{\sqrt{3}}N'(d_1)\right] > 0
\end{aligned}$$

2. *The vega of a put option on the assets studied is given by*

$$\begin{aligned}
\nu_P(S_T) &= S_0\sqrt{T}N'(d_1) > 0 \\
\nu_P(G_T) &= e^{-rT}E(G_T)\sqrt{T}\left[\frac{1}{6}\sigma\sqrt{T}N(-d_1) + \frac{1}{\sqrt{3}}N'(d_1)\right] > 0 \\
\nu_P\left(\frac{S_T}{G_T}\right) &= e^{-rT}E\left(\frac{S_T}{G_T}\right)\sqrt{T}\left[\frac{1}{6}\sigma\sqrt{T}N(-d_1) + \frac{1}{\sqrt{3}}N'(d_1)\right] > 0 \\
\nu_P\left(\frac{G_T}{S_T}\right) &= e^{-rT}E\left(\frac{G_T}{S_T}\right)\sqrt{T}\left[-\frac{5}{6}\sigma\sqrt{T}N(-d_1) + \frac{1}{\sqrt{3}}N'(d_1)\right]
\end{aligned}$$

Hence, the vega of call options on G_T or S_T/G_T is negative iff

$$\frac{N'(d_1)}{\sigma N(d_1)} < \frac{1}{2}\sqrt{\frac{T}{3}}$$

It is also clear that

$$\nu_P\left(\frac{G_T}{S_T}\right) < 0 \Leftrightarrow \frac{N'(d_1)}{\sigma N(-d_1)} < \frac{5}{2}\sqrt{\frac{T}{3}}$$

As shown in Figure 4, both inequalities can hold for reasonable parameter values. This figure exhibits geometric Australian option prices as a function of volatility. The averages are computed with infinite monitoring dates. The exercise price are $K = 0.8$ and $K = 1.1$ for calls and puts, respectively. The remaining parameters are: $r = 0.1, q = 0.03, T = 0.1$.

[Insert Figure 4 about here]

We see that the price of the call option on S_T/G_T first decreases and then increases with volatility. Its vega is zero when $\sigma = 0.67$. As expected, the price of the call option on G_T/S_T always increases with σ. The same is true for a put option on S_T/G_T. However, the price of the put option on G_T/S_T first decreases and then increases with volatility. Its vega reaches zero for $\sigma = 0.36$.

To summarize the results, both call and put geometric Australian option prices can increase and decrease with either time to maturity or volatility. This is clearly seen in Figure 5, that shows option prices as a function of both variables. The parameter values are: $r = 0.1, q = 0.03, n = \infty$. The exercise price is $K = 0.8$ for calls and $K = 1.1$ for puts.

[Insert Figure 5 about here]

Finally, Tables 1 and 2 present geometric option prices when $n = \infty$. Note that the option on S_n/G_n is equivalent to the option on G_n. This happens because we are taking $S_0 = 1$. We see that option prices for continuous averages are almost identical to those for discrete averages with 1,000 monitoring dates. For example, from Table 2 we have that when $\sigma = 0.2, T = 1, K = 1.0$ and $n = \infty$, the put options on $G_n, S_n/G_n$, and G_n/S_n have values of 2.935, 2.935, and 5.002, respectively, while that when $n = 1,000$ those prices are 2.937, 2.933, and 5.000, respectively.

4 Arithmetic Australian Options

We define the discrete arithmetic mean of the n stock prices $S_1, \cdots, S_n$ as

$$A_n = \frac{1}{n}\sum_{i=1}^{n} S_i, \quad A_0 \equiv S_0 \tag{19}$$

The continuous counterpart is given by

$$A_T = \frac{1}{T}\int_0^T S_t \, dt \tag{20}$$

As mentioned previously, the distribution of A_n is unknown. Therefore, we can not apply Proposition 1 to price options. As described in the following sections, two ways to overcome this problem are:

- To approximate the true distribution with an alternative one.
- To approximate the distribution of A_n with that of A_T.

4.1 Pricing the Options with the Edgeworth / Wilkinson Approximation

To price options, we approximate the risk-neutral distribution of the underlying asset at maturity with a tractable distribution. We perform this approximation by expanding the true distribution around the approximating one. This approach is called generalized Edgeworth series expansion. The coefficients of this expansion are function of the moments of the true and approximating distribution. Considering up to four terms in this expansion and specifying the approximating distribution to be lognormal, we will show that the (approximate) option price is equal to the Black-Scholes price plus three adjustment terms. These terms depend, respectively, on the difference between the variance, skewness, and kurtosis of the true and the lognormal distribution. The intuition is that the first four moments of the distribution are enough to reflect the effects of the distribution on option prices.

More concretely, we approximate the true probability distribution, $F(s)$, with an approximating distribution, $A(s)$. It is assumed that both distributions have continuous density functions, $f(s)$ and $a(s)$. We employ the following notation:

$$\begin{aligned}
\alpha_j(F) &= \int_{-\infty}^{\infty} s^j f(s)ds \\
\mu_j(F) &= \int_{-\infty}^{\infty} (s - \alpha_1(F))^j f(s)ds \\
\Psi(F,t) &= \int_{-\infty}^{\infty} e^{its} f(s)ds, \quad i = \sqrt{-1}
\end{aligned}$$

where $\alpha_j(F)$ and $\mu_j(F)$ are, respectively, the j-th non-central and central moments of F and $\Psi(F,t)$ is the characteristic function of F.[10]

Following Stuart and Ord (1987), the cumulants $k_j(F)$ of the distribution F are defined by the identity in t

$$\ln \Psi(F,t) = \sum_{j=1}^{\infty} k_j(F) \frac{(it)^j}{j!}$$

For practical purposes, we only need the first four cumulants in the Edgeworth series expansion. These cumulants are, respectively, the mean, the variance, the coefficient of skewness and the excess of kurtosis:

$$\begin{aligned} k_1(F) &= \alpha_1(F), \quad k_2(F) = \mu_2(F) \\ k_3(F) &= \mu_3(F), \quad k_4(F) = \mu_4(F) - 3\mu_2^2(F) \end{aligned}$$

Jarrow and Rudd (1982) prove the following series expansion for $f(s)$ around $a(s)$:

$$\begin{aligned} f(s) &= a(s) + \frac{k_2(F) - k_2(A)}{2!} \frac{d^2 a(s)}{ds^2} - \frac{k_3(F) - k_3(A)}{3!} \frac{d^3 a(s)}{ds^3} \\ &\quad + \frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!} \frac{d^4 a(s)}{ds^4} + \varepsilon(s) \end{aligned} \tag{21}$$

where, by construction, $k_1(F)$ is set equal to $k_1(A)$.

The difference between $f(s)$ and $a(s)$ depends on the cumulants of both distributions with weighting factors given by the derivatives of $a(s)$. The terms on the right-hand side of (21) reflect any difference in variance, skewness and kurtosis and variance between $f(s)$ and $a(s)$. The residual error, $\varepsilon(s)$, includes any remaining difference. For a numerical analysis of this error term, see Section 5 in Jarrow and Rudd (1982).

Now, we employ (21) to obtain an approximate option price. Using $f(s)$ as the true distribution of the asset price at maturity, we obtain the expected value at maturity of an option on this asset. Then, this expansion provides an approximated expected value for the option at maturity in terms of the approximating distribution, $a(s)$.

In a risk-neutral world, the true price of the call option, $C(F)$, is obtained by discounting its expected value at the risk-free rate:

$$C(F) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\}\, dF(S_T)$$

Using (21) and a little algebra, the call price becomes

$$\begin{aligned} C(F) &= C(A) + e^{-rT}\frac{k_2(F) - k_2(A)}{2!}a(K) - e^{-rT}\frac{k_3(F) - k_3(A)}{3!}\left.\frac{da}{dS_T}\right|_K \\ &\quad + e^{-rT}\frac{k_4(F) - k_4(A) + 3(k_2(F) - k_2(A))^2}{4!}\left.\frac{d^2a}{dS_T^2}\right|_K + \varepsilon(K) \qquad (22) \end{aligned}$$

where

$$C(A) = e^{-rT} \int_{-\infty}^{\infty} \max\{S_T - K, 0\}\, dA(S_T)$$

Although (21) is valid for any approximating distribution $a(s)$, a natural candidate is the lognormal one. In this case, (22) shows that the true option price is equal to the Black-Scholes price plus three adjustment terms.

As mentioned in the introduction, the Wilkinson approximation is a particular case of the Edgeworth expansion, where just the first two cumulants are used.

4.2 Pricing the Options with the Gamma Distribution

It is known that the infinite sum of lognormal distributions is a reciprocal gamma distribution. Using this distribution as state-price density function, Milevsky and Posner (1998) obtain a closed-form expression for the price of arithmetic Asians options. The solution is the same as the Black-Scholes formula where the normal distribution is replaced by the gamma one.

We briefly summarize the main characteristics of the gamma distribution. Let X be gamma distributed with parameters α and β, that is, $X \sim \Gamma(\alpha, \beta)$. Its density function is given by

$$g(x) = \frac{\beta^{-\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\}}{\Gamma(\alpha)}, \ x > 0$$

where $\Gamma(x)$ is the gamma function, defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

The mean and variance of the gamma distribution are

$$E(X) = \alpha\beta, \quad V(X) = \alpha\beta^2$$

If we define $Y = \frac{1}{X}$, then Y follows a reciprocal gamma distribution. Its first two non-central moments are

$$\begin{aligned} M_1 &= E(Y) = \frac{1}{\beta(\alpha-1)} \\ M_2 &= E(Y^2) = \frac{1}{\beta^2(\alpha-1)(\alpha-2)} \end{aligned}$$

The variance is given by

$$V(Y) = M_2 - M_1^2 = \frac{1}{\beta^2(\alpha-1)^2(\alpha-2)}$$

It is straightforward to obtain the following relationships:

$$\alpha = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \beta = \frac{M_2 - M_1^2}{M_1 M_2} \tag{23}$$

Hence, to price option, we must obtain the first two risk-neutral moments (M_1, M_2) of the underlying asset at maturity. Then, we compute α and β using (23). Finally, we use the cumulative density function of the gamma distribution as $N(.)$ in the Black-Scholes formula.

We next compute the moments of the arithmetic mean and the ratios.

Lemma 3

1. *The moments of the variable A_n are given by*

$$
\begin{aligned}
E(A_n) &= \frac{S_0}{n}\, h_1(r-q) && (24)\\
Cov(A_n, S_n) &= \frac{S_0}{n} E(S_n)[h_1(r-q+\sigma^2) - h_1(r-q)] && (25)\\
V(A_n) &= \left(\frac{S_0}{n}\right)^2 [2f_1(r-q+\sigma^2)[h_1(2(r-q)+\sigma^2) - h_1(r-q)] \\
&\quad - h_1(2(r-q)+\sigma^2) - [h_1(r-q)]^2] && (26)
\end{aligned}
$$

2. *The moments of the variable S_n/A_n, $n \geq 2$ can be approximated by*

$$
\begin{aligned}
E\left(\frac{S_n}{A_n}\right) &\simeq \frac{E(S_n)}{E(A_n)} - \frac{1}{(E(A_n))^2} Cov(A_n, S_n) + \frac{E(S_n)}{(E(A_n))^3} V(A_n) \\
V\left(\frac{S_n}{A_n}\right) &\simeq \left(\frac{E(S_n)}{E(A_n)}\right)^2 \left(\frac{V(S_n)}{(E(S_n))^2} + \frac{V(A_n)}{(E(A_n))^2} - 2\frac{Cov(A_n, S_n)}{E(S_n)E(A_n)}\right)
\end{aligned}
$$

with $E(A_n)$, $Cov(A_n, S_n)$ and $V(A_n)$ as given by (24)-(26).

3. *The moments of the variable A_n/S_n, $n \geq 2$ are given by*

$$
\begin{aligned}
E\left(\frac{A_n}{S_n}\right) &= \frac{1}{n} \exp\{-n(r-q-\sigma^2)\Delta t\}\, h_1(r-q-\sigma^2) && (27)\\
V\left(\frac{A_n}{S_n}\right) &= \left(\frac{1}{n}\right)^2 \exp\{-n(2(r-q)-3\sigma^2)\Delta t\} \\
&\quad \times [2f_1(r-q-\sigma^2)[h_1(2(r-q)-3\sigma^2) - h_1(r-q-2\sigma^2)] \\
&\quad - h_1(2(r-q)-3\sigma^2) - \exp\{-n\sigma^2\Delta t\} h_1^2(r-q-\sigma^2)] && (28)
\end{aligned}
$$

with

$$
\begin{aligned}
h_1(x) &= \sum_{i=1}^{n} e^{xi\Delta t} = f_1(x)\left(e^{xn\Delta t} - 1\right),\ x \neq 0, \quad h_1(0) = n && (29)\\
f_1(x) &= \frac{e^{x\Delta t}}{e^{x\Delta t} - 1},\ x \neq 0 && (30)
\end{aligned}
$$

Proof: See the Appendix. $\square$

Remark 1 *Several particular cases can be highlighted:*

1. *If $r = q - \sigma^2$, the moments of the variable A_n are given by*

$$\begin{aligned} E(A_n) &= \frac{S_0}{n}\, h_1(-\sigma^2) \\ Cov(A_n, S_n) &= \frac{S_0}{n} E(S_n)[n - h_1(-\sigma^2)] \\ V(A_n) &= 2\left(\frac{S_0}{n}\right)^2 f_1(\sigma^2) e^{-(n+1)\sigma^2\Delta t} \sum_{i=1}^{n} \Big(\cosh(\sigma^2 i\Delta t) - 1\Big) \end{aligned}$$

2. *If $r = q + \sigma^2$, the moments of the variable A_n/S_n, $n \geq 2$ are given by*

$$\begin{aligned} E\left(\frac{A_n}{S_n}\right) &= 1 \\ V\left(\frac{A_n}{S_n}\right) &= \left(\frac{1}{n}\right)^2 \Big[(2f_1(\sigma^2) - 1)\Big(e^{-\sigma^2\Delta t} h_1(\sigma^2) - n\Big) - n(n-1)\Big] \end{aligned}$$

Proof: See the Appendix. $\square$

Lemma 4

1. *The moments of the variable A_T are given by*

$$E(A_T) = \frac{S_0}{T}\Phi(r - q) \tag{31}$$

$$Cov(A_T, S_T) = \frac{S_0}{T} E(S_T)[\Phi(r - q + \sigma^2) - \Phi(r - q)] \tag{32}$$

$$V(A_T) = \left(\frac{S_0}{T}\right)^2 \left[2\frac{\Phi(2(r-q) + \sigma^2) - \Phi(r - q)}{r - q + \sigma^2} - (\Phi(r - q))^2\right] \tag{33}$$

2. *The moments of the variable S_T/A_T can be approximated by*

$$
\begin{aligned}
E\left(\frac{S_T}{A_T}\right) &\simeq \frac{E(S_T)}{E(A_T)} - \frac{1}{(E(A_T))^2} Cov(A_T, S_T) + \frac{E(S_T)}{(E(A_T))^3} V(A_T) \\
V\left(\frac{S_T}{A_T}\right) &\simeq \left(\frac{E(S_T)}{E(A_T)}\right)^2 \left(\frac{V(S_T)}{(E(S_T))^2} + \frac{V(A_T)}{(E(A_T))^2} - 2\frac{Cov(A_T, S_T)}{E(S_T)E(A_T)}\right)
\end{aligned}
$$

with $E(A_T)$, $Cov(A_T, S_T)$ and $V(A_T)$ as given by (31)-(33).

3. *The moments of the variable A_T/S_T are given by*

$$
E\left(\frac{A_T}{S_T}\right) = \frac{1}{T}\Phi(\sigma^2 - (r-q)) \tag{34}
$$

$$
\begin{aligned}
V\left(\frac{A_T}{S_T}\right) &= \left(\frac{1}{T}\right)^2 \exp\{-(2(r-q) - 3\sigma^2)T\} \\
&\quad \times \left[2\frac{\Phi(2(r-q) - 3\sigma^2) - \Phi(r-q-2\sigma^2)}{r-q-\sigma^2} \right. \\
&\quad \left. - \exp\{-\sigma^2 T\}\Phi^2(r-q-\sigma^2)\right]
\end{aligned} \tag{35}
$$

Proof: It is similar to that of Lemma 3 and it is omitted. □

Remark 2 *Several particular cases can be highlighted:*

1. *If $r = q - \sigma^2$, the moments of the variable A_T are given by*

$$
\begin{aligned}
E(A_T) &= \frac{S_0}{T}\Phi(-\sigma^2) \\
Cov(A_T, S_T) &= \frac{S_0}{T}E(S_T)[T - \Phi(-\sigma^2)] \\
V(A_T) &= 2\left(\frac{S_0}{T}\right)^2 \frac{e^{-\sigma^2 T}}{\sigma^4}(\sinh(\sigma^2 T) - \sigma^2 T)
\end{aligned}
$$

2. *If $r = q + \sigma^2$, the moments of the variable A_T/S_T are given by*

$$
\begin{aligned}
E\left(\frac{A_T}{S_T}\right) &= 1 \\
V\left(\frac{A_T}{S_T}\right) &= \left(\frac{1}{T}\right)^2 \left[2\,\frac{\Phi(\sigma^2) - T}{\sigma^2} - T^2\right]
\end{aligned}
$$

with $\Phi(.)$ as given by (55).

Proof: See the Appendix. □

Tables 3 and 4 show arithmetic call and put option prices (multiplied by 100) for different monitoring dates. The interest rate is 10% and the stock dividend yield is 3%. We price options on $A_n, S_n/A_n$ and A_n/S_n with three methods: Monte Carlo simulation,[11] Wilkinson approximation, and gamma distribution.

[Insert Tables 3 and 4 about here]

In the tables, we see that derivative prices with the three methods are very close. For example, in Table 3 we have that when $\sigma = 0.20, T = 0.5, K = 0.8$, and $n = 1,000$, the values of call options on A_n/S_n are 18.319, 18.324, and 18.321, respectively. Thus, Edgeworth expansions do not seem to be needed.

To price options on S_n/A_n with both the Wilkinson approximation and the gamma distribution, we have computed its moments using the approximation of Mood *et al* (1974) (see Lemma 10 in the Appendix). In the tables we see that those prices are very similar to the ones obtained with Monte Carlo, so that the approximations seem to work pretty well. For example, in Table 3 we see that when $\sigma = 0.2, T = 0.5, K = 0.8$, and $n = 1,000$, the values of call options using the Wilkinson approximation and the gamma distribution are 20.375 and 20.374, respectively, while the value obtained with Monte Carlo simulation is 20.377.

When the average is computed in continuous time (number of monitoring dates $= \infty$) we cannot use Monte Carlo simulation. However, as mentioned before, using 1,000 monitoring dates produces option prices very similar to those using continuous average. In Table 4 we see that the values of put options on S_n/A_n using the Wilkinson approximation and the gamma distribution are 8.863 and 8.887, respectively, for both 1,000 and ∞ monitoring dates.

To understand better why the three method produce very similar results, we plot the risk-neutral probability density function of the arithmetic stock price average in Figure 6.

[Insert Figure 6 about here]

The parameter values are: $r = 0.1,\ q = 0, \sigma = 0.2, T = 1, S_0 = 100$ and $n = \infty$. The expected value of the average price is 105.17, and the variance 152.74. For $n = \infty$ the true density function is reciprocal gamma, with parameters $\alpha = 74.42$ and $\beta = 1.29$E-4. This function is approximated with a lognormal distribution with the same moments. The density function is also estimated with Monte Carlo simulation, using a set of 50 runs of 10,000 paths with 1,000 time steps. We see that, for the parameter values used, the density functions are remarkably similar, hence the price of options on arithmetic stock prices must be close.

5 Conclusions

Australian options are options on the ratio of the stock price to its average or viceversa. They show up in variable purchase options, recently studied by Handley (2000, 2003).

If the stock price follows a geometric Brownian motion and the average is defined on geometric basis, these ratios also follow a geometric Brownian motion. Thus, we are able to obtain closed-form expressions for the price of the options. However, when the average is defined on arithmetic basis, the risk-neutral distributions of these ratios at maturity are unknown. Hence, to price the options we use a particular case of Edgeworth expansion (known as Wilkinson approximation) as well as a gamma approximation (following Milevsky and Posner (1998)). We compare the results with those obtained with Monte Carlo simulations, and we find that option prices are very similar in the three cases. Hence, in practice, it does not seem to be necessary to use high order moments in the Edgeworth expansion nor to require a large number of monitoring dates in the gamma approximation for pricing these claims.

Appendix

The following Lemma specifies several features of the lognormal distribution that will be useful to find later results:

Lemma 5

1. *Let* $Y = \ln(X)$ *be a normal random variable with mean* m *and variance* s^2*. Then,* X *follows a lognormal distribution, that is,* $X \sim \Lambda(m, s^2)$*. Its density function is given by*

$$f(x) = \frac{1}{sx\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln x - m}{s}\right)^2\right\}, \ x > 0 \qquad (36)$$

   Moreover, it is verified that

$$\begin{aligned} E(X) &= \exp\left\{m + \frac{1}{2}s^2\right\} && (37) \\ V(X) &= [E(X)]^2 \left[e^{s^2} - 1\right] && (38) \\ E(X^{-1}) &= \exp\{-2m\}\ E(X) && (39) \\ V(X^{-1}) &= [E(X^{-1})]^2 \left[e^{s^2} - 1\right] && (40) \end{aligned}$$

2. *The expectation of the truncated lognormal variable*

$$\tilde{X} = \begin{cases} X & \text{if } X \geq K \\ 0 & \text{if } X < K \end{cases}, \qquad K \in \mathbb{R}^+$$

   is given by

$$E(\tilde{X}) = E(X)\ N(s - D),\ D = \frac{\ln K - m}{s}$$

   where $N(.)$ *denotes the distribution function of a standard normal variable.*

Proof:

1. For proving expressions (37)-(38), see Johnson and Kotz (1970), p. 115. As $X^{-1} = e^{-Y}$, application of (37)-(38) leads to (39)-(40).

2. $E(\tilde{X})$ is obtained using (36) and a little of algebra. □

Proof of Proposition 1

The option payoff can be split in two components, the "contingent exercise payment" and the "contingent receipt of the stock". The payoffs for these claims are, respectively,

$$C_1(Z,T,T.K) = \begin{cases} -K & \text{if} \quad Z_T \geq K \\ 0 & \text{if} \quad Z_T < K \end{cases} \quad , \quad C_2(Z,T,T,K) = \begin{cases} Z_T & \text{if} \quad Z_T \geq K \\ 0 & \text{if} \quad Z_T < K \end{cases}$$

Then, the two components of the option are given by

$$C_1(Z,0,T,K) = E\left[e^{-rT}C_1(Z,T,T,K) \mid F_t\right] = -Ke^{-rT}P(Z_T \geq K) \quad (41)$$

$$C_2(Z,0,T,K) = E\left[e^{-rT}C_2(Z,T,T,K) \mid F_t\right] = e^{-rT}E[Z_T \mid Z_T \geq K] \quad (42)$$

Equation (4) implies that

$$P(Z_T \geq K) = N(d_2), \quad d_2 = \frac{\ln(e^{-\alpha_Z T}E(Z_T)/K) + \left(\alpha_Z - \frac{1}{2}\sigma_Z^2\right)T}{\sigma_Z\sqrt{T}} \quad (43)$$

Part 2 in Lemma 5 and a little algebra leads to

$$E[Z_T \mid Z_T \geq K] = E(Z_T)N(d_1), \quad d_1 = d_2 + \sigma_Z\sqrt{T} \quad (44)$$

Including (43)-(44) into (41)-(42) provides the final expression for the option price. □

We now state two lemmas that will be useful to obtain the moments of the geometric price average and its associated ratios, for both discrete and continuous monitoring.

Lemma 6 *Given the Brownian motions W_{t_i}, $i = 1, \cdots, n$, it is verified that*

$$
\begin{aligned}
V\left(\sum_{i=1}^{n} W_{t_i}\right) &= \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3}\,\Delta t \\
V\left(n\,W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right) &= \frac{(n-1)\left(n-\frac{1}{2}\right)n}{3}\,\Delta t
\end{aligned}
$$

Proof:

For $n \geq 2$, we have

$$
\begin{aligned}
V\left(\sum_{i=1}^{n} W_{t_i}\right) &= V\left(\sum_{i=1}^{n-1} W_{t_i} + W_{t_n}\right) = V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + t_n + 2\sum_{i=1}^{n-1} \mathrm{Cov}(W_{t_i}, W_{t_n}) \\
&= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + n\Delta t + 2\sum_{i=1}^{n-1} i\Delta t = V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + n^2\Delta t
\end{aligned}
$$

By induction, we get

$$
V\left(\sum_{i=1}^{n} W_{t_i}\right) = \sum_{i=1}^{n} i^2\,\Delta t = \frac{(n+1)\left(n+\frac{1}{2}\right)n}{3}\,\Delta t
$$

$$
\begin{aligned}
V\left(n\,W_{t_n} - \sum_{i=1}^{n} W_{t_i}\right) &= V\left(\sum_{i=1}^{n-1}[W_{t_n} - W_{t_i}]\right) \\
&= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) + \sum_{i=1}^{n-1}\sum_{i=1}^{n-1} \mathrm{Cov}(W_{t_n}, W_{t_n} - 2W_{t_i}) \\
&= V\left(\sum_{i=1}^{n-1} W_{t_i}\right) = \sum_{i=1}^{n-1} i^2\,\Delta t = \frac{n\left(n-\frac{1}{2}\right)(n-1)}{3}\,\Delta t
\end{aligned}
$$

□

Lemma 7 *Given the Brownian motions W_t, $t \in [0,T]$, it is verified that*

$$V\left(\int_0^T W_t dt\right) = V\left(T\,W_T - \int_0^T W_t dt\right) = \frac{T^3}{3}$$

Proof:

Integrating by parts, we have

$$\int_0^T (T-t)dW_t = \int_0^T W_t dt, \quad \int_0^T t dW_t = TW_T - \int_0^T W_t dt$$

Then, we get

$$V\left(\int_0^T W_t dt\right) = V\left(\int_0^T (T-t)dW_t\right) = \int_0^T (T-t)^2 dt = -\frac{(T-t)^3}{3}\bigg|_0^T = \frac{T^3}{3}$$

$$V\left(TW_T - \int_0^T W_t dt\right) = V\left(\int_0^T t dW_t\right) = \int_0^T t^2 dt = \frac{t^3}{3}\bigg|_0^T = \frac{T^3}{3}$$

□

The following two lemmas are used to obtain the moments of the arithmetic price average (discrete and continuous) and its associated ratios.

Lemma 8 *For $i, j = 1, 2, \cdots, n$ and $k \in \mathbb{R}$, we have*

$$E(S_i^k) = S_0^k \exp\left\{k\left(r - q + \frac{k-1}{2}\sigma^2\right) i\Delta t\right\} \tag{45}$$

$$E(S_i S_j^k) = E(S_i)E(S_j^k)\exp\{k\sigma^2 \min\{i,j\}\Delta t\} \tag{46}$$

$$E(S_i S_j S_n^k) = E(S_i S_j)E(S_n^k)\exp\{k\sigma^2 (i+j)\Delta t\} \tag{47}$$

Moreover, for $k \in \mathbb{R}$, we have

$$E(A_n S_n^k) = \frac{S_0}{n} E(S_n^k)\, h_1(r^*) \tag{48}$$

$$E(A_n^2 S_n^k) = \left(\frac{S_0}{n}\right)^2 E(S_n^k)[2f_1(r^* + \sigma^2)(h_1(2r^* + \sigma^2) - h_1(r^*)) - h_1(2r^* + \sigma^2)] \tag{49}$$

where $r^* = r - q + k\sigma^2$ and $h_1(.)$ and $f_1(.)$ as given by (29)-(30).

Proof:

$E(S_i^k)$ is obtained applying (37) to (8).

Using (8), (45), and a little algebra, we have, for $a, b, k \in \mathbb{R}$,

$$E(S_i^a S_j^b S_n^k) = E(S_i^a)E(S_j^b)E(S_n^k) \exp\{[ab \min\{i,j\} + k(ai + bj)]\sigma^2 \Delta t\}$$

For $k = 0$, we have

$$E(S_i^a S_j^b) = E(S_i^a)E(S_j^b) \exp\{ab\sigma^2 \min\{i,j\}\Delta t\} \tag{50}$$

and, then,

$$E(S_i^a S_j^b S_n^k) = E(S_i^a S_j^b)E(S_n^k) \exp\{k\sigma^2(ai + bj)\Delta t\} \tag{51}$$

Using (50) with $a = 1, b = k$ and (51) with $a = b = 1$ provides $E(S_i S_j^k)$ and $E(S_i S_j S_n^k)$, respectively.

- **Mean of** $A_n S_n^k$: Apply (19), (45) for $k = 1$, and (46) for $j = n$.
- **Mean of** $A_n^2 S_n^k$: Applying (19) and (47), we get

$$E(A_n^2 S_n^k) = \left(\frac{1}{n}\right)^2 E(S_n^k) z_n^*, \quad z_n^* = \sum_{i,j=1}^{n} E(S_i S_j) e^{k\sigma^2(i+j)\Delta t} \tag{52}$$

After some algebra, we get the recurrence law

$$z_n^* = z_{n-1}^* + S_0^2 \left[2f_1(r^* + \sigma^2)\left(e^{(2r^*+\sigma^2)n\Delta t} - e^{r^* n\Delta t}\right) - e^{(2r^*+\sigma^2)n\Delta t}\right], \; z_0^* = 0$$

with $f_1(.)$ as given by (30).

Applying this recurrence law for different values of n, we obtain

$$z_n^* = S_0^2 \left[2f_1(r^* + \sigma^2)(h_1(2r^* + \sigma^2) - h_1(r^*)) - h_1(2r^* + \sigma^2)\right]$$

Plugging this expression into (52), we get the final value for $E(A_n^2 S_n^k)$.

□

Lemma 9 *For $k \in \mathbb{R}$, we have*

$$E(A_T S_T^k) = \frac{S_0}{T} E(S_T^k)\Phi(r^*) \tag{53}$$

$$E(A_T^2 S_T^k) = 2\left(\frac{S_0}{T}\right)^2 E(S_T^k)\frac{\Phi(2r^* + \sigma^2) - \Phi(r^*)}{r^* + \sigma^2} \tag{54}$$

with

$$\Phi(x) = \frac{\exp\{xT\} - 1}{x}, \ x \neq 0, \quad \Phi(0) = T \tag{55}$$

and r^* as given by Lemma 8.

Proof:

$E(A_T S_T^k)$ is obtained using (20) and applying (45) with $k = 1$ and (46) with $j = n$.

To compute $E(A_T^2 S_T^k)$, use (20) and apply (46) with $k = 1$ and (47). □

The following Lemma will be useful to compute the moments of ratios involving arithmetic average asset prices:

Lemma 10 *Let X and Y be two random variables. Then, it is verified that*

$$E\left(\frac{X}{Y}\right) \simeq \frac{E(X)}{E(Y)} - \frac{1}{(E(Y))^2}Cov(X,Y) + \frac{E(X)}{(E(Y))^3}V(Y)$$

$$V\left(\frac{X}{Y}\right) \simeq \left(\frac{E(X)}{E(Y)}\right)^2 \left(\frac{V(X)}{(E(X))^2} + \frac{V(Y)}{(E(Y))^2} - 2\frac{Cov(X,Y)}{E(X)E(Y)}\right)$$

Proof: See Mood *et al* (1974), p. 181. □

Proof of Lemma 3

1. Moments of the arithmetic average A_n

   (a) Mean of A_n:

   Apply (48) with $k = 0$.

(b) Covariance of A_n with S_n:

Apply (48) for $k = 1$ and (24).

(c) Variance of A_n:

Apply (49) for $k = 0$ and (24).

2. Moments of the variable S_n/A_n:

   Apply part 2 in Lemma 10 with $X = S_n,\ Y = A_n$.

3. Moments of the variable A_n/S_n

   (a) Mean of A_n/S_n:

   Apply (45) for $i = n, k = -1$ and (48) for $k = -1$.

   (b) Variance of A_n/S_n:

   Apply (45) for $i = n, k = -2$, (49) for $k = -2$ and (27). □

Proof of Remark 1

1. Replace $r = q - \sigma^2$ into (24)-(25) to obtain $E(A_n)$ and $\mathrm{Cov}(A_n, S_n)$.

   To get $V(A_n)$, we will need the following relationships, satisfied by $h_1(.)$ and $f_1(.)$ (see (29)-(30)):

$$h_1(-a) = \exp\{-(n+1)a\Delta t\}h_1(a) \quad (56)$$

$$f_1(-a) = -\exp\{-a\Delta t\}f_1(a) \quad (57)$$

$$1 + h_1(a) = f_1(-a)\left[1 - e^{(n+1)a\Delta t}\right] \quad (58)$$

$$f_1(b) = \exp\{(b-a)\Delta t\}\frac{e^{a\Delta t} - 1}{e^{b\Delta t} - 1}f_1(a) \quad (59)$$

   Looking at (26), we need to compute

$$f_1(r - q + \sigma^2)[h_1(2(r-q) + \sigma^2) - h_1(r-q)]$$

Defining $x = r - q + \sigma^2$ and using (59), this expression becomes

$$f_1(x)f_1(x-\sigma^2)\left[e^{x\Delta t}\frac{e^{(x-\sigma^2)\Delta t}-1}{e^{(2x-\sigma^2)\Delta t}-1}\left(e^{(2x-\sigma^2)n\Delta t}-1\right)-\left(e^{(2x-\sigma^2)\Delta t}-1\right)\right]$$

Taking limits when $x \to 0$ and applying (57) and some algebra, we obtain

$$f_1(\sigma^2)\left[h_1(-\sigma^2) - ne^{-(n+1)\sigma^2\Delta t}\right]$$

Replacing this result into (26) and using (56) and (58), we obtain

$$V(A_n) = \left(\frac{S_0}{n}\right)^2 f_1(\sigma^2)e^{-(n+1)\sigma^2\Delta t}[h_1(\sigma^2) - 2n + h_1(-\sigma^2)]$$

It can be seen that, as expected, this variance is positive since

$$h_1(\sigma^2) - 2n + h_1(-\sigma^2) = 2\sum_{i=1}^{n}\left(\cosh(\sigma^2 i\Delta t) - 1\right) > 0$$

2. Replace $r = q + \sigma^2$ into (27) and use $h_1(0) = n$ to obtain $E(A_n/S_n)$. To get $V(A_n/S_n)$, define $x = r - q - \sigma^2$ and apply a similar procedure as in part 1 of this remark. □

Proof of Remark 2

1. Replace $r = q - \sigma^2$ into (31)-(32) to obtain $E(A_T)$ and $\text{Cov}(A_T, S_T)$. Applying the L'Hôpital's rule and using the relationship $\Phi(-x) = e^{-xT}\Phi(x)$, we obtain

$$V(A_T) = \left(\frac{S_0}{T}\right)^2 \frac{e^{-\sigma^2 T}}{\sigma^2}[\Phi(\sigma^2) - 2T + \Phi(-\sigma^2)]$$

It can be seen that, as expected, this variance is positive since

$$\Phi(\sigma^2) - 2T + \Phi(-\sigma^2) = \frac{2}{\sigma^2}[\sinh(\sigma^2 T) - \sigma^2 T] > 0$$

2. Replace $r = q + \sigma^2$ into (34) and use $\Phi(0) = T$ to obtain $E(A_T/S_T)$. To compute $V(A_T/S_T)$, apply the L'Hôpital's rule. □

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Footnotes

1. As will be indicated later, the distribution of the continuous-time average is known, allowing us to obtain exact analytical expressions for option prices.
2. It may be worth noting that if these variables were perfectly correlated, then the average would be lognormal. Alternatively, if these variables are i.i.d., applying the central limit theorem, the distribution of the average would converge to the normal one.
3. Other examples are Dewynne and Wilmott (1995a, 1995b), He and Takahashi (1995-96), Zvan *et al* (1998) or Shreve and Večer (2000).
4. See also Haykov (1993), Corwin *et al* (1996) and Nielsen and Sandmann (1996).
5. Dufresne (1990) and Yor (1993) are examples of papers that deal with the gamma distribution.
6. For completeness, this table includes the geometric average.
7. Recall that these prices have been multiplied by 100.
8. For completeness, this table includes the stock price and its geometric average. The formula for the geometric Asian option was first derived by Kemna and Vorst (1990).
9. It can be shown that for σ large enough ($\sigma > \sqrt{6/5}\sqrt{2T\ln(S_0) + 3(r-q)}$), this expected value is higher than $E(S_T)$.
10. Analogous notation is employed for the approximating distribution A.
11. We use 10,000 simulations and antithetic variables to reduce standard errors.

Table 1. Geometric Call option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	0.8	S_n	22.576	22.576	22.576	22.576	22.576
			G_n	22.576	20.696	20.561	20.548	20.546
			S_n/G_n	19.025	20.400	20.531	20.545	20.546
			G_n/S_n	19.025	18.133	18.161	18.166	18.166
20	1	0.8	S_n	25.187	25.187	25.187	25.187	25.187
			G_n	25.187	21.361	21.084	21.056	21.053
			S_n/G_n	18.097	20.756	21.023	21.050	21.053
			G_n/S_n	18.097	16.491	16.580	16.593	16.594
40	0.5	0.8	S_n	24.801	24.801	24.801	24.801	24.801
			G_n	24.801	20.766	20.558	20.538	20.536
			S_n/G_n	19.025	20.332	20.514	20.534	20.536
			G_n/S_n	19.025	20.266	20.908	20.979	20.987
20	0.5	1.1	S_n	3.175	3.175	3.175	3.175	3.175
			G_n	3.175	0.905	0.737	0.721	0.719
			S_n/G_n	0	0.549	0.701	0.717	0.719
			G_n/S_n	0	0.282	0.374	0.384	0.385

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on S_n and G_n we take $S_0 = 1$. For options on G_n, derivative prices can be computed with the Merton's (1973) formula when $n = 1$, and with the Kemna and Vorst's (1990) formula when $n = \infty$.

Table 2. Geometric Put option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	1.0	S_n	3.930	3.930	3.930	3.930	3.930
			G_n	3.930	2.591	2.439	2.423	2.422
			S_n/G_n	0	2.250	2.405	2.420	2.422
			G_n/S_n	0	3.321	3.516	3.535	3.537
20	1	1.0	S_n	4.639	4.639	4.639	4.639	4.639
			G_n	4.639	3.135	2.955	2.937	2.935
			S_n/G_n	0	2.732	2.915	2.933	2.935
			G_n/S_n	0	4.724	4.976	5.000	5.002
40	0.5	1.0	S_n	9.277	9.277	9.277	9.277	9.277
			G_n	9.277	6.149	5.763	5.724	5.719
			S_n/G_n	0	5.283	5.676	5.715	5.719
			G_n/S_n	0	5.314	5.493	5.508	5.510
20	0.5	1.1	S_n	9.300	9.300	9.300	9.300	9.300
			G_n	9.300	8.753	8.712	8.714	8.713
			S_n/G_n	9.512	8.688	8.710	8.713	8.713
			G_n/S_n	9.512	10.691	10.759	10.765	10.766

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on S_n and G_n we take $S_0 = 1$. For options on G_n, derivative prices can be computed with the Merton's (1973) formula when $n = 1$, and with the Kemna and Vorst's (1990) formula when $n = \infty$.

Table 3. Arithmetic Call option prices.

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
20	0.5	0.8	A_n MC	22.551	20.737	20.731	20.723	-
			A_n W	22.576	20.885	20.729	20.714	20.712
			A_n GD	22.535	20.883	20.728	20.711	20.711
			S_n/A_n MC	19.025	20.329	20.381	20.377	-
			S_n/A_n W	19.025	20.209	20.360	20.375	20.377
			S_n/A_n GD	19.025	20.208	20.358	20.374	20.375
			A_n/S_n MC	19.025	18.332	18.317	18.319	-
			A_n/S_n W	19.025	18.389	18.330	18.324	18.324
			A_n/S_n GD	19.025	18.388	18.327	18.321	18.321
20	1	0.8	A_n MC	25.089	21.431	21.366	21.387	-
			A_n W	25.187	21.736	21.418	21.387	21.384
			A_n GD	25.059	21.716	21.404	21.373	21.370
			S_n/A_n MC	18.097	20.642	20.730	20.766	-
			S_n/A_n W	18.097	20.376	20.682	20.713	20.717
			S_n/A_n GD	18.097	20.366	20.667	20.698	20.701
			A_n/S_n MC	18.097	16.841	16.884	16.899	-
			A_n/S_n W	18.097	16.964	16.887	16.880	16.883
			A_n/S_n GD	18.097	16.947	16.862	16.855	16.857

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 3. Arithmetic Call option prices (cont.).

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
40	0.5	0.8	A_n MC	24.687	21.141	21.111	21.206	-
			A_n W	24.801	21.468	21.188	21.161	21.158
			A_n GD	24.400	21.358	21.099	21.074	21.071
			S_n/A_n MC	19.025	19.853	20.011	19.987	-
			S_n/A_n W	19.025	19.641	19.918	19.947	19.951
			S_n/A_n GD	19.025	19.562	19.819	19.846	19.849
			A_n/S_n MC	19.025	21.432	21.620	21.535	-
			A_n/S_n W	19.025	21.276	21.569	21.599	21.620
			A_n/S_n GD	19.025	21.213	21.489	21.517	21.535
20	0.5	1.1	A_n MC	3.141	0.802	0.756	0.757	-
			A_n W	3.176	0.933	0.778	0.761	0.759
			A_n GD	3.198	0.980	0.802	0.785	0.782
			S_n/A_n MC	0	0.593	0.612	0.682	-
			S_n/A_n W	0	0.528	0.681	0.698	0.699
			S_n/A_n GD	0	0.549	0.705	0.721	0.722
			A_n/S_n MC	0	0.389	0.404	0.413	-
			A_n/S_n W	0	0.302	0.387	0.396	0.401
			A_n/S_n GD	0	0.320	0.409	0.419	0.424

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 4. Arithmetic Put option prices.

Parameters			Asset	Number of monitoring dates (n)				
σ(%)	T	K	Z_n	1	10	100	1,000	∞
20	0.5	1.0	A_n MC	3.930	2.331	2.417	2.387	-
			A_n W	3.930	2.530	2.387	2.372	2.371
			A_n GD	3.868	2.511	2.371	2.357	2.356
			S_n/A_n MC	0	2.365	2.477	2.512	-
			S_n/A_n W	0	2.342	2.490	2.505	2.506
			S_n/A_n GD	0	2.330	2.475	2.491	2.494
			A_n/S_n MC	0	3.328	3.440	3.408	-
			A_n/S_n W	0	3.176	3.417	3.440	3.451
			A_n/S_n GD	0	3.175	3.415	3.440	3.449
20	1	1.0	A_n MC	4.639	2.821	2.771	2.872	-
			A_n W	4.639	3.038	2.875	2.859	2.857
			A_n GD	4.479	2.991	2.834	2.819	2.817
			S_n/A_n MC	0	3.043	2.994	3.055	-
			S_n/A_n W	0	2.913	3.086	3.103	3.105
			S_n/A_n GD	0	2.884	3.051	3.068	3.069
			A_n/S_n MC	0	4.736	4.854	4.817	-
			A_n/S_n W	0	4.436	4.775	4.809	4.833
			A_n/S_n GD	0	4.433	4.770	4.803	4.827

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

Table 4. Arithmetic Put option prices (cont.).

Parameters			Asset	Number of monitoring dates (n)				
$\sigma(\%)$	T	K	Z_n	1	10	100	1,000	∞
40	0.5	1.0	A_n MC	9.277	5.465	5.402	5.554	-
			A_n W	9.277	5.874	5.525	5.490	5.487
			A_n GD	8.971	5.792	5.457	5.423	5.419
			S_n/A_n MC	0	5.855	6.000	5.988	-
			S_n/A_n W	0	5.837	6.202	6.238	6.242
			S_n/A_n GD	0	5.794	6.147	6.182	6.186
			A_n/S_n MC	0	5.078	5.202	5.201	-
			A_n/S_n W	0	4.880	5.193	5.224	5.290
			A_n/S_n GD	0	4.822	5.124	5.154	5.217
20	0.5	1.1	A_n MC	9.300	8.571	8.581	8.576	-
			A_n W	9.300	8.613	8.589	8.588	8.588
			A_n GD	9.322	8.639	8.613	8.611	8.611
			S_n/A_n MC	9.512	8.838	8.827	8.829	-
			S_n/A_n W	9.512	8.858	8.862	8.863	8.863
			S_n/A_n GD	9.512	8.879	8.886	8.887	8.887
			A_n/S_n MC	9.512	10.586	10.626	10.617	-
			A_n/S_n W	9.512	10.453	10.603	10.618	10.623
			A_n/S_n GD	9.512	10.472	10.625	10.640	10.646

Prices are multiplied by 100. The interest rate is 10% and the dividend yield 3%. For options on A_n we take $S_0 = 1$. MC, W and GD refer to Monte Carlo simulation, Wilkinson approximation and gamma distribution, respectively.

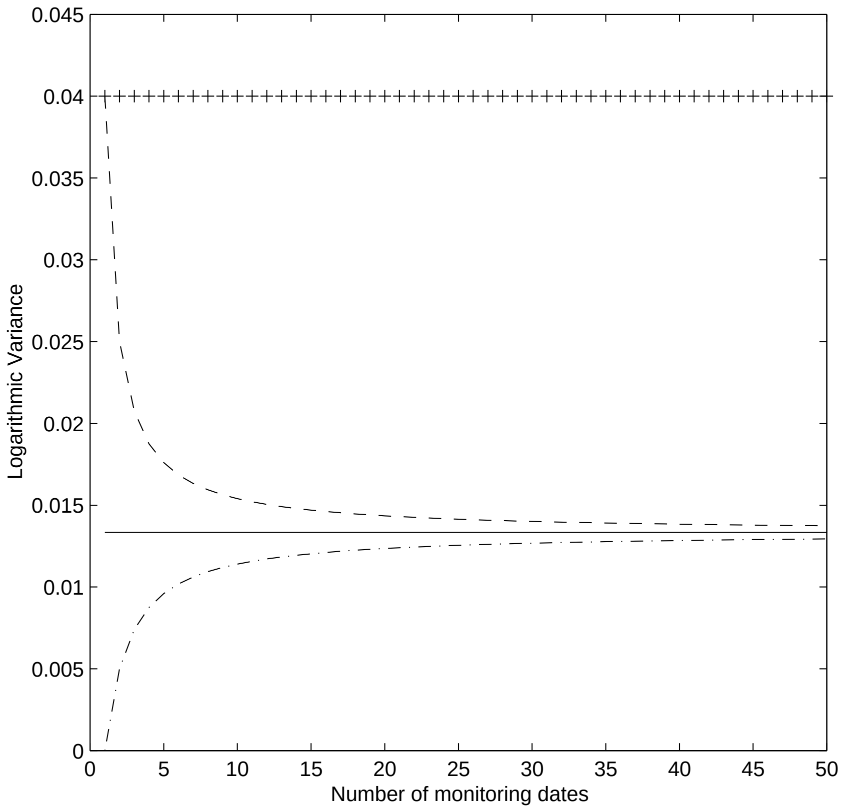

Figure 1: **Plot of the logarithmic variance ($\sigma_Z^2 T$) as function of** n. The parameter values are $\sigma = 0.2$ and $T = 1$. The figure depicts the logarithmic variance of the stock price (line marked with the sign "+"), G_n (dashed line), G_T (solid line), S_n/G_n and G_n/S_n (dotted-dashed line). These values are given by the following table:

Z_n	$\sigma_Z^2 T$
S_n	$\sigma^2 T$
G_n	$\frac{(n+1)\left(n+\frac{1}{2}\right)}{3n^2}\ \sigma^2 T$
S_n/G_n	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\ \sigma^2 T$
G_n/S_n	$\frac{(n-1)\left(n-\frac{1}{2}\right)}{3n^2}\ \sigma^2 T$

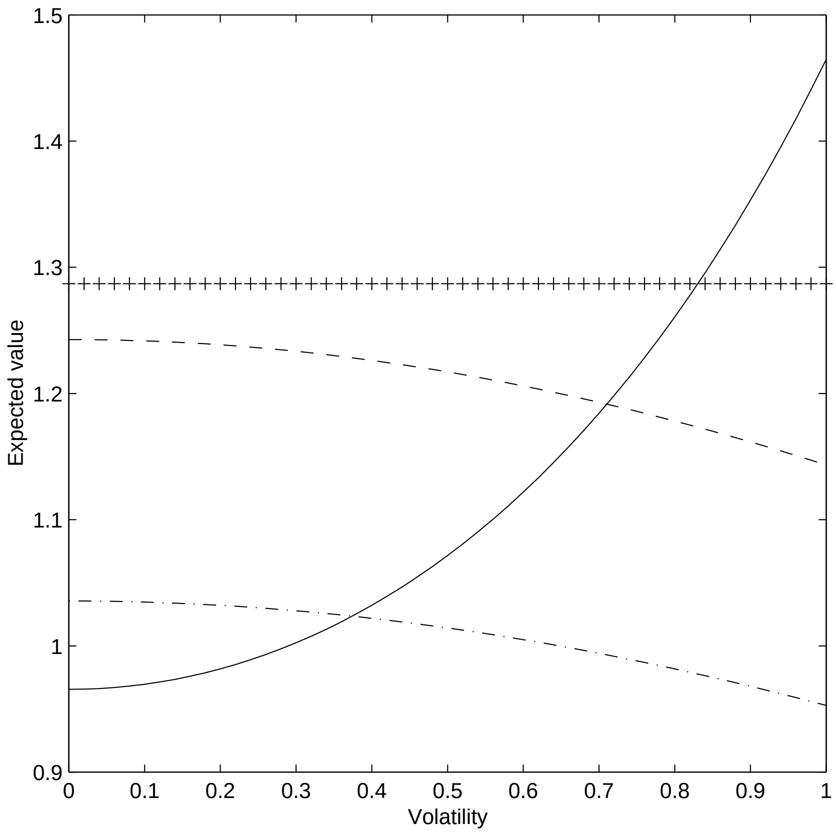

Figure 2: **Plot of the expected value $E(Z_T)$ as function of σ^2.** The parameter values are $r = 0.1, q = 0.03, \sigma = 0.2$ and $T = 1$. We assume $S_0 = 1.2$. The figure depicts the expected values of the stock price (line marked with the sign "+"), G_T (dashed line), S_T/G_T (dotted-dashed line), and G_T/S_T (solid line). These values are given by the following table:

Z_T	$E(Z_T)$
S_T	$S_0 \exp\{(r-q)T\}$
G_T	$S_0 \exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$
S_T/G_T	$\exp\left\{\frac{1}{2}\left(r-q-\frac{1}{6}\sigma^2\right)T\right\}$
G_T/S_T	$\exp\left\{-\frac{1}{2}\left(r-q-\frac{5}{6}\sigma^2\right)T\right\}$

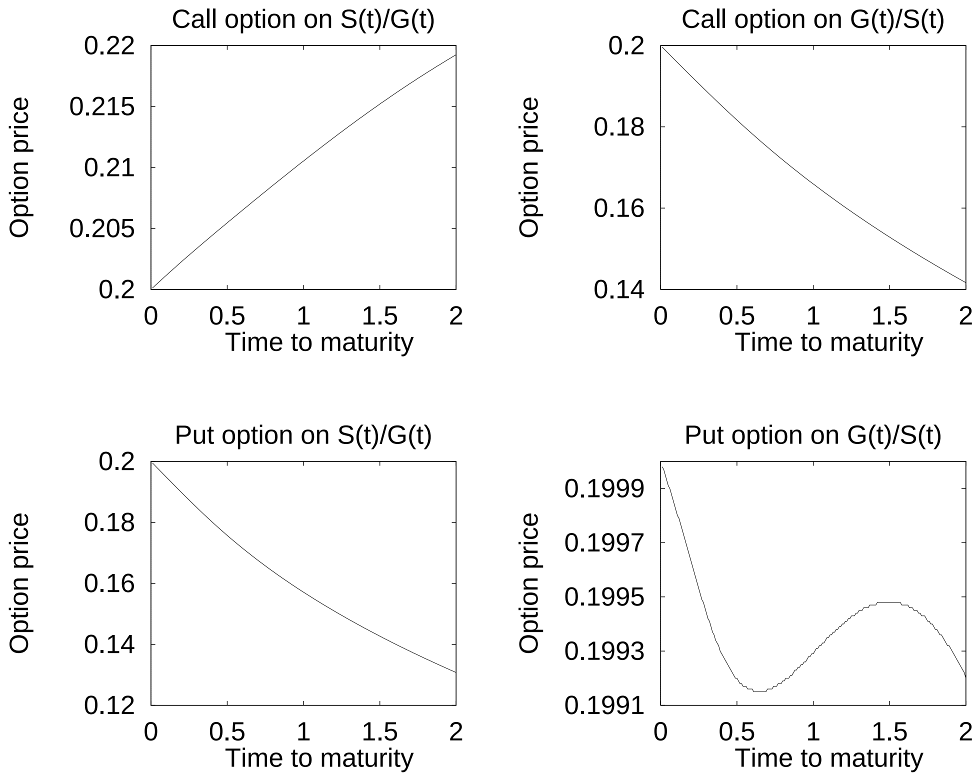

Figure 3: **Geometric Australian option prices as a function of time to maturity**. The exercise price is $K = 0.8$ for calls and $K = 1.2$ for puts. The other parameter values are: $r = 0.1, q = 0.03, \sigma = 0.2$, and $n = \infty$.

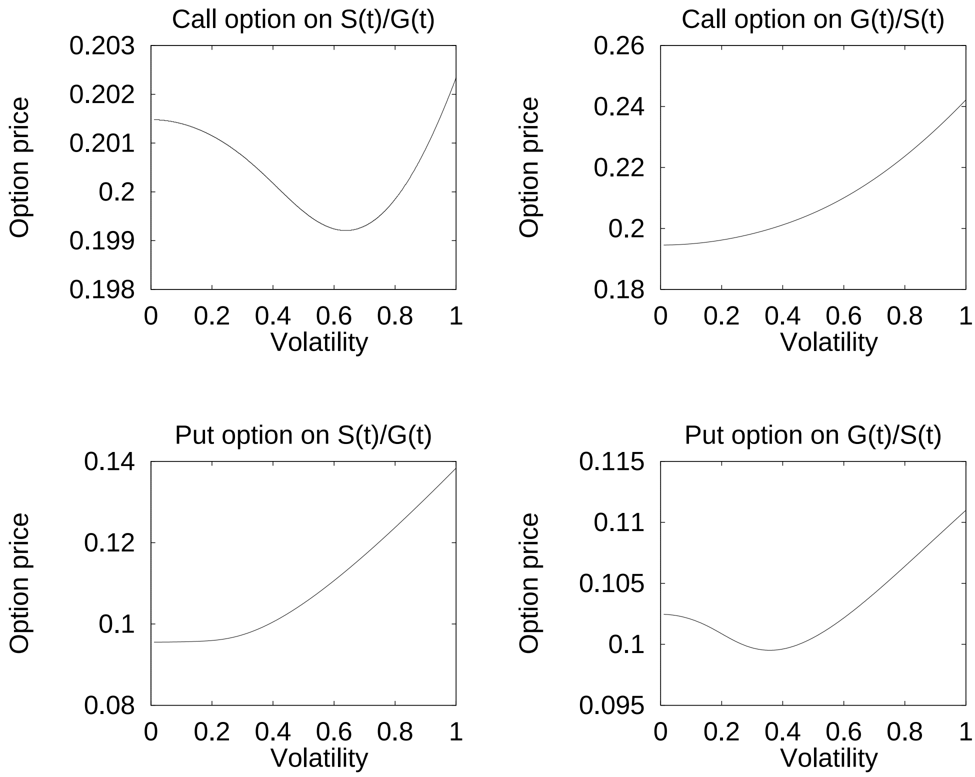

Figure 4: **Geometric Australian option prices as a function of volatility (σ).** The exercise price is $K = 0.8$ for calls and $K = 1.1$ for puts. The other parameter values are: $r = 0.1, q = 0.03, T = 0.1$, and $n = \infty$.

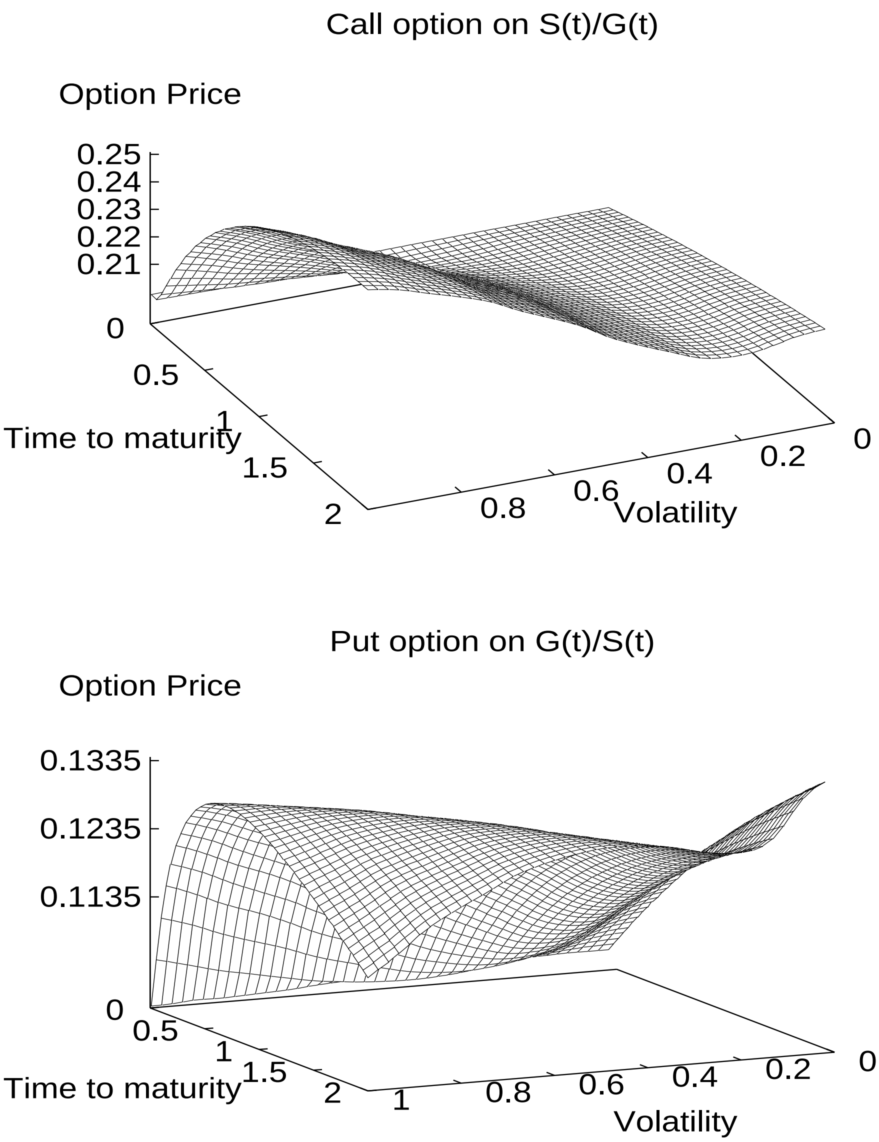

Figure 5: **Geometric Australian option prices as a function of time to maturity and volatility (σ).** The exercise price is $K = 0.8$ for the call option and $K = 1.1$ for the put option. The other parameter values are: $r = 0.1, q = 0.03$, and $n = \infty$.

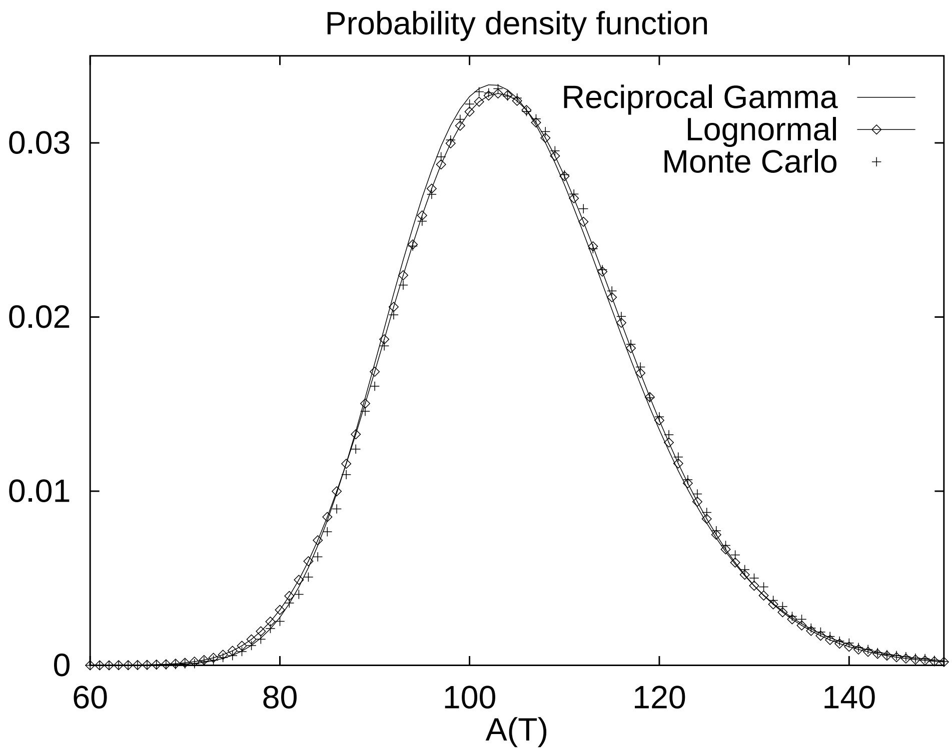

Figure 6: **Risk-neutral probability density function of the arithmetic stock price average at maturity**. The parameter values are: $r = 0.1, \; q = 0, \sigma = 0.2, T = 1, S_0 = 100$ and $n = \infty$. The expected value of the average price is 105.17, and the variance 152.74. For $n = \infty$ the true density function is reciprocal gamma, with parameters $\alpha = 74.42$ and $\beta = 1.29$E-4. This function is approximated with a lognormal distribution with the same moments. The density function is also estimated with Monte Carlo simulation, using a set of 50 runs of 10,000 paths with 1,000 time steps.