EXPECTATIONS, INSTITUTIONS, AND GROWTH*

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ABSTRACT

This paper studies a strategic model of growth in which firms' accumulation and technological decisions are subject to both friction and external effects. This gives rise to a wide multiplicity of equilibrium behavior, which is consistent with quite different performances of the economy (e.g., bounded vs. sustained growth.)

The focus of the paper is three-fold. First, it explores (and even characterizes in certain cases) the technological conditions underlying the different alternative possibilities. Second, it analyzes the relationship between optimality and equilibrium. Third, it proposes institutional arrangements (an optimal tax/subsidy scheme) which induces, at no "distortionary cost", the selection of the optimal equilibrium path.

KEYWORDS: Expectations, Institutions, Growth.
1.- INTRODUCTION

Paraphrasing Lucas (1993), one of primary objectives of the Theory of Growth and Development (some would even say the core of it) should be to provide insight into the occurrence of both "miracles" and "anti-miracles" among different countries. Nay, it should also help us understand why some given country may shift from the anti-miracle to the miracle "mode", or even viceversa, at different times of its history.

Such is the main motivation of the present paper. The model it proposes to this effect consists of the following basic components, some standard ((a) and (b) below), some not:

(a) Firms and consumers are involved in an intertemporal setup where consumers save, firms invest and produce, all of them adopting their decisions with perfect foresight on the future course of the process.

(b) At each point time, spot prices (in this case, only the interest rate) are determined to clear competitive markets.

(c) Firms may choose among several alternative technologies in order to carry out their production. These technological decisions are subject to friction, in the sense that they will only be able to revise them with some positive probability, possibly less than one.

(d) The production potential of any given technology is fixed. However, its actual productivity at any point in time depends on technological spillovers; specifically, on the fraction of other firms which use technologies of "comparable sophistication".

(e) The marginal productivity of any given technology is eventually small for large enough scales of production. Thus, sustained growth can only be fueled in our context by an appropriate (and never-ending) series of technological transitions to progressively "more advanced" technologies.

In a model with the above features it is not surprising that there co-exist alternative equilibrium paths. Questions of coordination should become crucial in determining which kind of growth path the economy will follow; in particular, whether the economy will experience sustained growth or instead will be mired in eventual stagnation. A detailed analysis of when this is
indeed a dilemma, and of what are the considerations (technological and otherwise) which underlie it, will be one of the first objectives of the paper. Specifically, we shall identify some conditions which induce (or even characterize in certain cases) the scenarios in which sustained growth is possible.

A second concern of the paper will be to relate equilibrium and optimal performance in our context. Defining the latter as a solution to the standard planner’s problem, we identify certain conditions under which it becomes supportable by equilibrium. These conditions, however, will be seen to be quite fragile. Thus, this state of affairs should be seen as an "ideal", only to be sufficiently approximated in general circumstances.

The third and final objective of the paper is to integrate the issues described in the former two paragraphs into an institutional (or "implementation") framework. Specifically, we shall ask the following question: Are there institutions (or mechanisms) which can "solve" the generally wide multiplicity of the model and select the optimal equilibrium? A clearcut answer in the affirmative is provided to this question in terms of a certain tax/subsidy scheme, under the assumption that the planner (or government) can exploit its relative size to commit to one such scheme. Indeed, it will be seen that the mere possibility of such a commitment allows the government to eschew any actual tax or subsidy in "shaping" the right expectations and thus implementing the desired performance.

I end this Introduction with some brief references to related research. Some of the main features of the model have already been discussed in the literature in the framework of alternative models. They are analyzed, however, in scenarios different from the present one (usually static), or they are not jointly integrated into a single framework.

Externalities and/or increasing returns were introduced in the growth literature by the seminal work of Romer (1986). Later on, Lucas (1988), Azariadis & Drazen (1990), or Grossman & Helpman (1991) have explored different formulations of them as the source of sustained growth.
The link between complementarities and coordination problems has been extensively discussed in recent literature, often with the purpose of providing foundations for Keynesian-type models. Pursuing the lead of Diamond (1982) and Cooper & John (1988), Murphy, Shleifer & Vishny (1989) and Matsuyama (1992) have investigated the connections between complementarities and growth.

When issues of coordination arise in an intertemporal context with friction, the resulting problem of equilibrium selection becomes a struggle between history versus expectations. The ensuing multiplicity of equilibria arising in such context has been studied by Weil (1989), Krugman (1991), Matsuyama (1991), or Young (1993).

Finally, we can find in Stokey (1988) or Lucas (1993) the notion that growth should be conceived as a recurrent process of shifting to more advanced goods (or technologies) when the former ones are exhausted. Related ideas have been explored in Young (1991).

The rest of the paper is organized as follows. First, in Section 2, I present the model. Section 3 states and discusses the results. Section 4 concludes with a summary. For the sake of smoothness in the presentation, the proofs of the results are relegated to an Appendix.

2.- THE MODEL

2.1. The agents

There are two population of agents: consumers and firms, each of them with the cardinality of the continuum. For simplicity, both of these populations are parameterized by the interval [0,1], a typical consumer indexed by \( i \in I = [0,1] \), and a typical firm indexed by \( j \in J = [0,1] \).

2.2. Technologies

Let \( \mathcal{T} \) denote the set of possible technologies. For the sake of focus, we shall abstract from the issue of technological innovation and assume that all
the technologies in $\mathcal{I}$ are available to every firm $j \in F$ at any point in time. Thus, in our context, the technological problem confronted by each firm simply boils down to finding the appropriate timing of its technological adoption decisions.

The technologies in $\mathcal{I}$ are indexed by the natural numbers.\(^1\) The productive potential of each technology $\rho_n \in \mathcal{I}$, $n \in \mathbb{N}$, is given by a corresponding "production function"

$$f_n: \mathbb{R}_+ \rightarrow \mathbb{R}_+,$$

and an associated fixed cost $\phi_n \geq 0$. The function $f_n(\cdot)$ maps the amount of (variable) capital $k$ used in production by a typical firm into its corresponding production potential $f_n(k)$. For simplicity, it is assumed that no capital is "consumed" in production (i.e., zero depreciation).

As explained below, $f_n(k)$ is to be interpreted as the production "ceiling" of technology $\rho_n$, which becomes fully achievable only when its users enjoy enough positive externalities from the current technological configuration of the economy. For simplicity, each $f_n(\cdot)$ will be assumed increasing, concave, and twice differentiable, with uniformly bounded first and second derivatives.\(^2\) Inaction will also be open to firms as a feasible "technological" choice. Formally, it will be identified with the technology $\rho_0$, whose production function satisfies $f_0(\cdot) \equiv 0$.

Technologies in $\mathcal{I}$ (now also including $\rho_0$) are assumed indexed according to their corresponding level of "sophistication". This will be particularly useful when formalizing, in a simple way, one of the crucial features of the model, namely, that technological externalities and complementarities are important in determining the productiveness of any given technology. This idea, under different forms, is quite fundamental in recent models of growth.

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\(^1\) The cardinality of the set $\mathcal{I}$ is irrelevant for our analysis. Its countability is only assumed for the sake of formal simplicity.

\(^2\) These assumptions can be substantially generalized at some notational cost. For example, concavity of each $f_n(\cdot)$ can be fully dispensed with, due to the usual concrification procedure allowed by the assumption of a continuum of firms.
and technological change (recall our previous discussion of the literature). In our context, it will be assumed that the extent of the production potential $f_n(\cdot)$ actually reachable by a some technology $\rho_n$ at a certain point in time depends, as specified below, on the current technology profile of the economy $\mu$, where

$$\mu: \mathcal{T} \rightarrow \mathbb{R}_+$$

is a density function over the set $\mathcal{T}$ that specifies corresponding shares of technology use across the economy.

Denote by $F_n(\mu, k)$ the actual production induced by technology $\rho_n$ and capital $k$ when the current technological profile of the economy is $\mu$. Endowing $\Delta(\mathcal{T})$, the set of densities on $\mathcal{T}$, with the supremum norm, each function $F_n(\cdot)$ will be assumed twice differentiable, with uniformly bounded first and second derivatives at points of "full potential", i.e., when $F_n(\mu, k) = f_n(k)$.

The following further conditions on $F(\cdot)$ are also contemplated.

$$\forall \rho_n \in \mathcal{T}, \forall \mu \in \Delta(\mathcal{T}), \forall k \geq 0,$$

(a) $0 \leq F_n(\mu, k) \leq f_n(k)$;

(b) $\sum_{s=n}^{\infty} \mu(\rho_s) = 1 \Rightarrow F_n(\mu, k) = f_n(k)$;

(c) $\sum_{s=0}^{n-1} \mu(\rho_s) = 1 \Rightarrow F_n(\mu, k) = 0$. (2)

Condition (a) simply says that the actual productivity of a technology is bounded by its "potential". On the other hand, Conditions (b) and (c) bear on the idea of technological externality which is central to our approach. They reflect the general idea that technologies are ordered according to a certain notion of progressive sophistication (or advancement), with technological spillovers flowing from the relatively advanced technologies to those that are of no higher sophistication level. The particular formulation of these spillovers is left largely unspecified. Only the two extreme configurations need to be explicitly considered. Specifically, (b) states that if almost all firms in the economy are at least "as advanced" as a firm using technology $\rho_n$, then this firm can enjoy the full potential of this technology. On the other hand, (c) postulates that, if almost all firms in the economy use a technology
less advanced than \( \rho_n \), then the firm’s actual productivity with this technology is very low (zero, for simplicity).

The conditions on the technological scenario formalized by (2) above represent the essential (and indispensable) core of the model. However, in order to fix ideas and also facilitate some of our analysis, it will be useful to incorporate two further postulates: first, the natural idea that the more sophisticated a technology is, the higher its production potential, at least for high enough scales of production; second, that more sophisticated technologies also exhibit higher fixed costs. These two postulates are formalized as follows:

\[
\forall n, n' \in \mathbb{N}, \ n' > n, \ \exists z \in \mathbb{R}_+; \ \forall z = \phi_n + k = \phi_{n'} + k', \ f_n'(k') \geq f_n(k) \iff z \geq z. \quad (3)
\]

\[
\forall n, n' \in \mathbb{N}, \ n' > n \implies \phi_{n'} > \phi_n. \quad (4)
\]

Condition (3) simply says that, for any two given technologies, there is a certain threshold of capital (fixed plus variable) which marks the scale at which the more "sophisticated" one starts to dominate the other. Condition (4), on the other hand, requires no further comment. As mentioned, these conditions are not crucial for our main results. They could be dispensed with at the cost of some increase in the complexity of our analysis.

2.3. Intertemporal Decision Problems

Time is measured discretely and indexed by \( t = 0, 1, 2, \ldots \). At any given \( \hat{t} \geq 1 \), all agents (both consumers and firms) are assumed to confront with certainty the ensuing time path of economy-wide variables which define their decision problems. Specifically, it is assumed that at each \( \hat{t} = 1, 2, \ldots \) all agents know the paths:

\[
\{r(t)\}_{t=\hat{t}}^{\infty} \] of current and future interest rates;

\[
\{\pi(t)\}_{t=\hat{t}}^{\infty} \] of current and future average profits;

\[
\{\mu(t)\}_{t=\hat{t}}^{\infty} \] of current and future technological profiles.

Given these paths as well as their corresponding individual "states" (see below), the decision problems of both consumers and firms are now described in turn.
2.3.1. Consumers

Consumers are assumed all identical, which will permit us to dispense with their respective index in what follows. Let

\[ U: \mathbb{R}_+ \rightarrow \mathbb{R} \]  \hspace{1cm} (6)

represent their "instantaneous" utility of consumption every period, which will is twice differentiable, strictly concave, and satisfies \( \lim_{c \to \infty} U'(c) = 0 \).

Denote by \( \delta \in (0,1) \) the subjective rate of time preference of each consumer. At any \( \hat{i} = 1,2,\ldots, \) every consumer is assumed to choose a consumption path \( \{c(t)\}_{t=\hat{i}}^{\infty} \) that, given some savings \( S(\hat{i}-1) \) derived from the preceding period, solves the following optimization problem \( P_e(\hat{i}) \):

\[
\text{Max} \quad \sum_{t=\hat{i}}^{\infty} \delta^{t-\hat{i}} U(c(t))
\]  \hspace{1cm} (6)

subject to:

\[
\sum_{t=\hat{i}}^{\infty} \frac{c(t)}{R(\hat{i},t)} \leq S(\hat{i}-1) + \sum_{t=1}^{\infty} \frac{\pi(t)}{R(\hat{i},t)},
\]  \hspace{1cm} (7)

where \( R(\hat{i},\hat{i}) = (1+r(\hat{i})) \), and for \( t > \hat{i}, \)

\[
R(\hat{i},t) = (1+r(\hat{i})) (1+r(\hat{i}+1)) \ldots (1+r(t)).
\]  \hspace{1cm} (8)

Note that the above formulation implicitly assumes that firms' profits are uniformly distributed among consumers. The savings available at \( t = 1, S(0) \), are a parameter of the model. As time proceeds, it is implicit in (7) that savings \( S(t) \) follows the law of motion:

\[
S(t) = S(t-1) \left(1 + r(t)\right) + \pi(t) - c(t).
\]  \hspace{1cm} (9)

The variable \( S(\hat{i}-1) \) represents a natural state (or "summary") of the process for the consumers' decision problem at any time \( \hat{i} \). Denoting by \( r^i \) and \( \pi^i \) the sequences of interest rates and average profits prevailing from any time \( \hat{i} \)
onwards, it will be convenient to rely on such state-variable nature of $S(i-1)$ and define the correspondence $G(S(i-1); \ r^i, p^i)$ as the set of consumption values $c(i)$ which solve the optimization problem $P(i)$ above.

2.3.2. Firms

At each $i = 1, 2, \ldots$, every firm $j \in J$ makes a two-fold decision.

First, it has to decide how much capital $k_j(t)$ to rent at the current interest rate $r(t)$. Thus, as in much of the theoretical growth literature, we make the convenient assumption that the investment decision of firms is fully flexible (i.e., capital is "putty-putty", the same single good used for both consumption and investment).

Second, each firm has to choose the technology to which its rented capital will be applied. Denote such decision by $d_j(t) \in \mathcal{T}$. One of the crucial features of our model is the assumption that this technological decision is subject to "friction". Specifically, we consider the following simple (time-invariant) formulation:

At each $t$, every firm $j \in J$ enjoys the same independent probability $p \in (0,1]$ of being able to revise its previous technological choice.

Thus, with the complementary independent probability $(1-p)$ each firm $j$ is rigidly attached to its preceding technology, i.e., $d_j(t) = d_j(t-1)$.

Several different interpretations can be provided for the proposed formulation. For example, it may be simply assumed that new technological information reaches firms in a random (but unbiased) fashion. Or, instead, it may be supposed that technological adjustment is linked to processes of labor (or machine) turnover which proceed in a stochastic and independent manner across firms. Alternative formulations for the general idea of "technological friction" could also be considered, which would nevertheless preserve the gist of our conclusions. (3)

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3 For example, it could be postulated that it is only with some given periodicity (say, every $n$ periods) that each firm can decide on a new production technology. Even though the "overlapping structure" induced by such an scenario has some theoretical advantages, it has the drawback of not being time-invariant.
A second consideration is involved in the firms' technological decision: we shall assume that it cannot be too advanced relative to the current technological base of the economy. To formalize this idea denote the "technological frontier" of the economy at $t$ by

$$
\bar{\rho}(t) = \max \left\{ \rho(t-1), \sup \{ n \in \mathbb{N} : \rho_n \in \text{supp} \mu(t-1) \} \right\}, \bar{\rho}(0) = \rho_0.
$$

(10)
i.e., the most sophisticated technology adopted with positive frequency in the past.

Associated to any such technology, we shall contemplate the set $\mathscr{F}(\bar{\rho}(t))$ which consists of all those technologies which can be adopted by firms at any given $t$ when the technological frontier of the economy is $\bar{\rho}(t)$. This set defines the technological choice set of all those firms which enjoy at $t$ the possibility of revising their current technology (see below). Only the following self-explanatory conditions are demanded from the correspondence $\mathscr{F}(\cdot)$:

(i) $\forall \rho_n \in \mathscr{I}, \rho_n \in \mathscr{F}(\rho_n), |\mathscr{F}(\rho_n)| < +\infty$;
(ii) $\forall \rho_n, \rho_{n'} \in \mathscr{I}, n' > n, \rho_{n'} \in \mathscr{F}(\rho_n') \Rightarrow \rho_n \in \mathscr{F}(\rho_n').

(11)

Hence, we may write $\mathscr{F}(\rho_n) = \{ \rho_{n'} \in \mathscr{I} : n' \leq \hat{n} \}$ for some $\hat{n} \geq n$.

Given the above described scenario, the decision problem of firms may be formalized as follows. Define the individual state $\omega_j(\hat{t})$ of firm $j$ at $\hat{t}$ as the tuple $(d_j(\hat{t}-1), \theta_j(\hat{t}), \bar{\rho}(\hat{t}))$ where, as above, $d_j(\hat{t}-1)$ stands for the preceding technological choice of firm $j$, the second component establishes whether firm $j$ has the opportunity of technological revision at $\hat{t}$ (indicated by $\theta_j(\hat{t}) = 1$) or not ($\theta_j(\hat{t}) = 0$), and the third specifies the current technological base of the economy (common to all firms).

The optimization problem $P_j(\hat{t})$ solved by a typical firm $j$ at $\hat{t}$ can be decomposed into two type of contingencies (or states):

(i) First, suppose that $d_j(\hat{t}-1) = \rho_n$ and $\theta_j(\hat{t}) = 0$. Then, its decision problem is to choose the $k_j(\hat{t})$ that solves the following optimization program:
Max $F_n(\mu(\hat{i}), k_j(\hat{i})) - r(\hat{i}) (k_j(\hat{i}) + \phi_n)$. (4) (12)

For future reference, for each technology $\rho_n \in \mathcal{F}$, we shall denote by $k_n^*(\mu, r)$ the set of maximizers of the above expression as a function of the prevailing profile $\mu$ and the interest rate $r$. Correspondingly, $\pi_n^*(\mu, r)$ will stand for the maximum of the above expression.

(ii) Secondly, suppose that $\theta_j(\hat{i}) = 1$. Then, firm $j$ can change its former technology and thus is assumed to choose any $\rho_n \in \mathcal{F}(\hat{\rho}(\hat{i}))$ that solves:

$$\text{Max } \sum_{t=\hat{i}}^{\infty} \frac{(1-p)^{t-\hat{i}}}{R(\hat{i}, t)} \pi_n^*(\mu(t), r(t)),$$ (13)

Of course, once chosen some particular $\rho_n$ that solves (13) the implicit assumption in this expression is that the firm will choose some element in $k_n^*(\mu(t), r(t))$ for all $t \geq \hat{i}$ at which it has not yet enjoyed the option of revising its current technology. As before, we shall find it useful to have some notation to refer to the set of maximizers of (13). It will be denoted by $\rho^*(\hat{i}, \bar{\mu}^{\hat{i}-1})$, where $\bar{r}^i$ and $\bar{\mu}^{\hat{i}-1}$ stand for the (assumed known) sequences of interest rates and profiles prevailing from $\hat{i}$ and $\hat{i}-1$, respectively, onwards. (Note that the knowledge of $\mu(\hat{i}-1)$ is enough to determine $\bar{\rho}(\hat{i})$.)

The objective function postulated in (12) and (13) is the natural one to consider here, given the intertemporal budget constraint confronted in (7) by consumers, also assumed the owners of the firms. In order to consolidate the two cases considered above into a single formulation, we shall denote by $H(d_j(\hat{i}-1), \theta(t); \bar{r}^i, \bar{\mu}^{\hat{i}-1})$ the set of optimal decision pairs $(k, \rho_n)$ which are induced by (i) or (ii), depending on the corresponding value of $\theta$, in the obvious fashion. The induced correspondence $H(\cdot)$ is the firms' counterpart of the correspondence $G(\cdot)$ which summarized the decision problem of consumers in Subsection 2.3.1.

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4 For simplicity, the fixed cost is conceived as a capital fixed requirement whose price is the current interest rate. This is inessential for the analysis, which would remain unchanged if the fixed cost were conceived instead as a requirement of consumption good.
2.4. Equilibrium

As standard in intertemporal rational-expectations models of market interaction, an equilibrium must incorporate the three following considerations: (i) agents’ decisions have to be optimal, as described in the previous subsection, given a certain prediction for the future path of economy-wide variables; (ii) markets must clear every period; (iii) the path predicted for the economy-wide variables has to result from (i.e., be consistent with) the agents’ decisions and market clearing postulated in (i) and (ii).

To avoid technical measurability issues, it will be convenient to focus on those agents’ strategies that are symmetric. That is, those strategies which depend only on each agent’s state and possibly on time, but not on the agent’s identity.

As explained above, we shall identify the consumers’ state space $\Omega_c$ with the set of possible "savings" currently available. Thus, we make $\Omega_c = \mathbb{R}$, where of course negative savings indicate indebtedness.

On the other hand, for every firm $j \in J$, its identical state space $\Omega_f$ is identified with the set $\mathcal{F} \times \{0,1\} \times \mathcal{F}$, whose generic element $(\rho_n, \theta, \tilde{\rho})$ indicates the status-quo technology $\rho_n$ used by firm $j$, whether the firm can or cannot revise its status-quo technology ($\theta = 0, 1$), and the current technological frontier $\tilde{\rho}$.

Each agent’s state $\omega_a$, $a \in \{c, f\}$, determines, in particular, her current choice set $\Gamma(\omega_a)$. Thus, for firms, $\Gamma(\rho_n, \theta, \tilde{\rho}) = \mathbb{R}_+ \times \mathcal{F}(\tilde{\rho})$ if $\theta = 1$ (that is, the firm can choose any amount of capital and any technology) or $\Gamma(\rho_n, \theta, \tilde{\rho}) = \mathbb{R}_+ \times \{\rho_n\}$ if $\theta = 0$. For consumers, on the other hand, $\Gamma(\omega_c) = \mathbb{R}_+$ for any $\omega_c \in \Omega_c$ since current consumption is not restricted due to the possibility of indebtedness.

We are now in a position to introduce the notion of symmetric strategy. Denote $\Gamma^c = \mathbb{R}_+$ and $\Gamma^f = \mathbb{R}_+ \times \mathcal{F}$ and let $\mathcal{B}(\Gamma^c)$ represent the set of Borel
probability measures on the set $\Gamma^a$. A *symmetric strategy* for an agent of type $a \in \{c,f\}$ is a sequence of mappings $\{\sigma^a_i\}_{t=1,2,\ldots} = \sigma^a_i: \Omega_a \to \mathcal{B}(\Gamma^a)$,

\begin{equation}
\text{such that } \text{supp } \sigma^a_i(\omega_a) \subseteq \Gamma(\omega_a).
\end{equation}

Assume that our large-number context induces no aggregate uncertainty on the evolution of economy-wide variables and these can be identified with their corresponding expected values. (See Judd (1985) or Feldman & Gilles (1985) for the technical issues involved in this assumption.) Under this assumption, given a firm strategy $\{\sigma^f_i\}_{t=1,2,\ldots}$, it is immediate to construct the following functions for all $t = 1,2,\ldots$:

- the transition function for the technological profile:

\begin{equation}
\mu(t) = \mathcal{M}(\mu(t-1),\sigma^f_t);
\end{equation}

- the average demand of capital by firms:

\begin{equation}
K(t) = \mathcal{K}(\mu(t-1),\sigma^f_t);
\end{equation}

- the average profit of firms:

\begin{equation}
\pi(t) = \mathcal{P}(\mu(t-1),\sigma^f_t, r(t)).
\end{equation}

A formal definition of equilibrium now follows.

**Definition 1**: Given $\mu(0)$ and $S(0)$, an *equilibrium* consists of a pair of strategies $\varrho = [(\sigma^c_i)_{t=0}^\infty, (\sigma^f_i)_{t=1}^\infty]$ and a tuple of time sequences $\chi = [(c(t))_{t=0}^\infty, (S(t))_{t=0}^\infty, (\mu(t))_{t=0}^\infty, (K(t))_{t=1}^\infty, (\pi(t))_{t=1}^\infty, (r(t))_{t=1}^\infty]$ such that:

(a) $\forall t \geq 1, c(t) \in \text{supp}[\sigma^c_i(S(t-1))] \subseteq G(S(t-1); \rho, \pi^f);$

(b) $\forall t \geq 1, \mu(t)(\rho_n) > 0 \Rightarrow \text{supp}[\sigma^f_i(\rho_n, 0)] \subseteq H(\rho_n, 0; \rho, \mu(t-1)), \forall \theta \in \{0,1\};$

(c) $\{S(t)\}_{t=0}^\infty, \{\mu(t)\}_{t=0}^\infty, \{K(t)\}_{t=1}^\infty$, and $\{\pi(t)\}_{t=1}^\infty$ are determined by, respectively, (9), (15), (16), and (17);

(d) $\forall t \geq 1, S(t-1) = K(t).$
The interpretation of the above conditions is straightforward. Conditions (a) and (b) simply state that agents' strategies are optimal for almost everyone (in particular, for those firms whose technology is adopted with positive frequency), given their predictions of the future paths of the economy-wide variables. Condition (c) requires that such predictions are accurate. Finally, (d) demands that the capital market clears every period.

3.- ANALYSIS

The issues addressed in this section are grouped into the following three topics: Existence (Subsection 3.1), Sustained Growth (Subsection 3.2), and Optimality, Expectations, and Equilibrium Selection (Subsection 3.3).

3.1. Existence

A basic existence result obtains from the following two assumptions.

(A.1) \( \forall p_n \in \mathcal{T}, \lim_{k \to \infty} f_n'(k) = 0. \) (5)

(A.2) \( |\text{supp } \mu(0)| < \infty. \) (6)

The first assumption postulates that the marginal returns vanish in the limit, as the amount of capital used with any given technology grows. This is a familiar assumption in Classical Growth Theory, often included as part of the so-called Inada Conditions. Assumption (A.2), on the other hand, is essentially technical, assuming that only a finite number of technologies are represented in the initial distribution.

**Theorem 1:** Assume (A.1) and (A.2). An equilibrium exists.

**Proof:** See the Appendix.

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5 As standard, \( h'(\cdot) \) and \( h''(\cdot) \) will denote first and second derivatives of any given function \( h(\cdot) \) all throughout the paper.

6 The notation \( |\cdot| \) stands for the cardinality of the set in question.
Theorem 1 represents a basic existence result, on which we now build the rest of our analysis.

3.2. Sustained Growth

One of our main purposes will be to show that, in the context described, sustained growth is attainable under appropriate conditions. By (A.1), such indefinite growth will require, in particular, that firms recurrently shift to ever more "advanced" technologies along the equilibrium. However, due to the externalities contemplated by (2), only if a large enough "critical mass" of firms plan to make such a shift in the future may any given firm find it profitable to do so itself. Combining both observations, it is clear that, in our context, bounded growth always remains an equilibrium possibility. Moreover, our next result also establishes that it becomes the only possible equilibrium outcome if "friction" is large (i.e., if \( p \) is quite small), and the following strengthening of (A.1) holds:

\[
(A.1)' \quad \forall p_n \in \mathcal{S}, \exists M_n \in \mathbb{R}_+ \text{ s.t. } \forall k \in \mathbb{R}_+, f_n(k) \leq M_n.
\]

**Proposition 1:** Assume (A.1) and (A.2). There is always an equilibrium \((\sigma, \lambda)\) whose consumption path \(\{c(t)\}_{t=0}^{\infty}\) satisfies \(\limsup_{t \to \infty} c(t) < +\infty\).

Suppose, moreover, that \((A.1)'\) also holds. Then, \(\exists \bar{p} > 0\) such that if \(p \leq \bar{p}\), every equilibrium yields a bounded consumption path.

**Proof:** See the Appendix.

Even if friction is small, and a large fraction of firms could adjust their technology simultaneously, sustained growth at equilibrium will generally require much more. In particular, any contemplated technological transition along the process could appear unsupportable at equilibrium given what would be the predicted market reaction to it (e.g., given some upward pressure on the interest rate predicted after any such transition). As we shall see, only under certain conditions will the technological transitions needed for sustained growth be consistent with equilibrium.
The analysis dealing with these issues is structured as follows. First, Theorem 2 provides a "technological" condition under which, even for small friction, the growth of the economy is bounded at equilibrium. Reciprocally, of course, the violation of this condition (a reverse inequality) can be read as a necessary condition for sustained growth. Next, Theorem 3 proposes a sufficient "separating" condition across technologies which ensures the possibility of sustained growth, again under small enough friction. (As indicated by Proposition 1, only a scenario with sufficiently low friction will yield sustained growth and rich technological dynamics.) Finally, Theorem 4 presents some benchmark circumstances for which the necessary condition for sustained growth induced by Theorem 2 also becomes "almost sufficient" for (the possibility of) it.

Before stating formally our results, I introduce some new notation which will be required for the remaining of the paper. First, for each \( \rho \in \mathcal{F} \), let

\[
Y_n = \{ y = (x, k + \phi_n): x \leq f_n(k) \}.
\]  

stand for its "production possibility set", with \( \mathcal{Y} = \bigcup_{n \in \mathbb{N}} Y_n \) representing the production possibility set across all technologies. The upper envelope of the set \( \mathcal{Y} \) is denoted by the function \( \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) where, for all \( z \in \mathbb{R}_+ \), \( \psi(z) = \sup \{ x: (x, z) \in \mathcal{Y} \} \). The convex hull of \( \mathcal{Y} \) will be denoted by \( \hat{\mathcal{Y}} \), with corresponding upper envelope (the concavification of \( \psi(\cdot) \)) given by \( \hat{\psi}(z) = \sup \{ x: (x, z) \in \hat{\mathcal{Y}} \} \). Both \( \psi(\cdot) \) and \( \hat{\psi}(\cdot) \) will be assumed well-defined, i.e., finite. Finally, notice the obvious point that if all functions \( f_n(\cdot) \) are differentiable, so is \( \hat{\psi}(\cdot) \).

Let \( \beta = \frac{1-\delta}{\delta} \). The first of the theorems advanced above follows.

**Theorem 2:** Let \( \lim_{k \to \infty} \psi'(k) < \beta \) \( \exists \rho < 1 \) s.t. if \( p \equiv \rho \), every equilibrium \( (\sigma, \mathcal{X}) \) induces a consumption path \( \{c(t)\}_{t=0}^{\infty} \) which satisfies

\[
\lim_{t \to \infty} \sup_{t \to \infty} c(t) < \infty.
\]

\[\text{Note that the limit on } \psi'(\cdot) \text{ is always well defined because of the concavity of } \hat{\psi}(\cdot).\]
Proof: See the Appendix.

The preceding result establishes a technological condition under which, for small enough friction, sustained growth is impossible at equilibrium. It expresses the intuitive idea that if the marginal productivity along the upper envelope of the "global" production possibility set of the economy eventually falls below $\beta = (1-\delta)/\delta$, accumulation and growth must come to a standstill. For, in this case, any marginal unit of consumption good saved at $t$ for future production is transformed in less that $1/\delta (= 1+\beta)$ units of consumption at $t+1$. Since $1/\delta$ expresses the rate at which future marginal utility is discounted over present one, no further accumulation can proceed at equilibrium in this case. This condition is analogous to that applicable in standard growth models, where a similar condition on the (fixed) production function of the economy also implies bounded growth at equilibrium.

Of course, Theorem 2 above can be easily reformulated in a reciprocal manner to express a necessary condition for sustained growth. Complementary to such necessary condition, our next concern will be to find sufficient conditions for sustained growth. With this in mind, consider the following statement.

(S) There exist $\eta_1, \eta_2, \eta_3 > 0$ such that $\forall p_n \in \mathcal{F}, \exists p_n^* \in \mathcal{F}(p_n)$:

\[ \forall k \in \mathbb{R}_+, \eta_1 \leq \beta - f_n'(k) \leq 0 \Rightarrow \]

(i) $\forall (x,z) \in Y_n^+, n' < n$, $\forall f_n(k) \cdot (f_n'(k), k+\phi_n) \geq \nabla f_n(k) \cdot (x,z) + \eta_2$;  \( ^{8} \)

(ii) $f_n''(k) \geq f_n'(k) + \eta_3$.

Verbally, (S) says that for every $p_n$ acting as the technological frontier of the economy there must exist some other reachable (i.e., within $\mathcal{F}(p_n)$) higher-index technology $p_n^*$ which satisfies the following condition:

If the marginal productivity along $p_n$ becomes close enough (from above) to the critical value $\beta$, then a general shift to $p_n^*$ entails, for all firms, higher profits (part (i)), as well as a higher marginal productivity (part (ii)).

---

\( ^{8} \) As standard, the notation $\nabla f_n(k)$ stands for the gradient $(1, f_n'(k))$ of the function $f_n^{(\cdot)}$ at the point $(k, f_n'(k))$. 

20
The graphical interpretation of this condition is straightforward (cf. Figure 1). Part (ii) simply involves comparing the slopes of two technologies at a certain range of points; Part (i), on the other hand, embodies a separation of the point \( \left( f_n(k), k + \Phi_n \right) \) from all technology sets \( Y_n', n' \leq n, \) through the hyperplane defined by the gradient \( \nabla f_n(k) \).

![Graphical representation of technology comparison](image)

**FIGURE 1**

The economic interpretation of (S) is also quite clear. It essentially requires that when any technology is close to its "maturation" (i.e., near the point where its marginal productivity is so low that any further accumulation stops - recall above) there must always exist an alternative technology to which an "instantaneous" transition is profitable at equilibrium (part (i)), and with a marginal productivity high enough that it permits further accumulation (part (ii)).

It is important to understand in this respect the role played by the discount factor \( \delta \). Very crucially, this parameter affects the extent to which a given technology may expand. In general, only if this process of maturation
may proceed far enough, will it be possible for some ensuing technological transition to take place. (This is certainly the case in terms of Condition (S), but is also generally true, as exemplified by Theorem 4 below.)

To press the former point even further, consider a technological scenario where (A.1) holds and higher-index technologies always enjoy a marginal productivity higher than the lower-index ones at any given point. Then, it is immediate to check that (S) must necessarily apply in the limit (possibly for \( \eta_1 \) and \( \eta_2 \) equal to zero) as the discount rate \( \delta \) converges to one. This exercise merely points to the fact that, in general, the possibility or impossibility of sustained growth in our context may be linked to the intertemporal trade-offs contemplated by the agents of the economy. In particular, only those economies which are not too impatient will be able to let the process of technological maturation proceed to the point where technological transitions are feasible.

**Theorem 3:** Assume (A.1), (A.2) and (S). \( \exists p < 1 \) such that if \( p \approx p \), there is an equilibrium \( (\sigma, \chi) \) whose consumption path \( \{c(t)\}_{t=0}^{\infty} \) satisfies

\[
\lim_{t \to \infty} c(t) = 0.
\]

**Proof:** See the Appendix.

To fix ideas, it may be useful to discuss briefly a particular example. This will also serve to confirm the consistency of our assumptions and condition (S), thus ensuring the non-voidness of the results.

**Example:**

Consider a family of technologies \( \mathcal{S} \) parameterized by \( \{q_{n}, \phi_{n}\}_{n \in \mathbb{N}} \) with \( \{\phi_{n}\}_{n \in \mathbb{N}} \) satisfying (4) and \( \{q_{n}\}_{n \in \mathbb{N}} \) inducing a corresponding family of production functions of the form

\[
f_{n}(k) = q_{n} \ln (1+k)
\]

(19)

A natural scenario to explore is one where the productivity parameters \( q_{n} \) grow at a given rate \( v > 0 \), i.e., where for all \( n, \bar{n} \in \mathbb{N} \),

\[
\frac{q_{\bar{n}}}{q_{n}} = (1+v)^{(\bar{n}-n)},
\]

(20)
with $q_1 > 1$. (9) Clearly, given (20), the different properties of the environment (in particular, whether or not condition (S) holds) must hinge upon the behavior of $\{\phi_n\}_{n \in \mathbb{N}}$. If, compared to the rate $v$, this latter sequence grows too fast, then condition (S) will fail. Nay, even the condition of Theorem 2 (much stronger than the negation of (S)) will hold if fixed costs grow very fast, thus forcing the economy to bounded growth in every equilibrium.

To assess these issues, it is useful to focus on the benchmark case (cf. Figure 2) where the sequence of fixed costs $\{\phi_n\}_{n \in \mathbb{N}}$ defines an upper envelope of $\hat{\mathcal{J}}$ of the form

$$\hat{\psi}(z) = a + bz,$$  \hspace{1cm} (21)

for some pre-established $b > q_1$, with $a$ determined by the arbitrarily chosen $\phi_1 \geq 0$.

---

9 This concrete formulation is inconsistent with the uniform bounds on first- and second-order derivatives postulated above. Whereas I choose this example due to its simplicity, it should be clear, however, how it could be readily modified to have such uniform bound prevail.
The whole sequence of fixed costs \( \{ \phi_n \}_{n \in \mathbb{N}} \) is determined jointly with the corresponding set \( \{ x_n \}_{n \in \mathbb{N}} \) of "tangency points" through the following equations, for all \( n = 1, 2, \ldots, \)

\[
q_1 \frac{(1+v)^{n-1}}{1 + x_n - \phi_n} = b ; \tag{22a}
\]

\[
q_1(1+v)^{n-1} \ln (1+x_n - \phi_n) - bx_n = q_1(1+v)^n \ln (1+x_{n+1} - \phi_{n+1}) - bx_{n+1}. \tag{22b}
\]

Substituting, for each \( n = 1, 2, \ldots, \) the corresponding term \((1+x_n - \phi_n)\) into their respective equation in (22b), it is possible to solve for the sequence \( \{ x_n \}_{n \in \mathbb{N}} \). After immediate algebraic manipulations, we obtain the following:

\[
x_{n+1} = \frac{q_1}{b} (1+v)^{n-1} \left[ v \left( \frac{q_1}{b} \right) + (1+vn) \ln (1+v) \right] + x_n, \tag{23}
\]

where \( x_1 \) is given by the chosen value of \( \alpha \). Relying on the fact that:

\[
x_n = (1+v)^{n-1} \frac{q_1}{b} - \phi_n - 1, \tag{24}
\]

we may compute the sequence \( \{ \phi_n \}_{n \in \mathbb{N}} \) as follows:

\[
\phi_{n+1} = \frac{q_1}{b} (1+v)^{n-1} \left[ v \left( \frac{q_1}{b} - 1 \right) + (1+vn) \ln (1+v) \right] + \phi_n, \tag{25}
\]

with \( \phi_1 = (q_1/b) - x_1 - 1 \). If \( b \) is chosen greater than \( \beta \), Theorem 2 leaves open the possibility that there exists some equilibrium with sustained growth. (That is, the sufficient condition for bounded growth stated by this theorem is not met by a context with \( b > \beta \).) Indeed, under certain particular conditions which are specified in Theorem 4 below, some equilibrium with sustained growth is always shown to exist if the latter inequality is satisfied.

However, for the moment, let us focus only on condition (S), which Theorem 3 has shown to be sufficient for (the possibility of) sustained growth at equilibrium, provided friction is small. Part (ii) of Condition (S) is
trivially met by our context. As for its Part (i), it should be clear that a necessary and sufficient requirement for it to hold is that the slope of each function $f_n' \cdot \cdot \cdot$ at the "tangency point" $x_{n+1}$ corresponding to the subsequent function $f_{n+1}' \cdot \cdot \cdot$ be bounded above $\beta$. That is:

$$\exists \eta > 0: \forall n = 1,2,\ldots, f'_n(x_{n+1}) \leq \beta + \eta.$$  \hspace{1cm} (26)

Using (24) we can compute:

$$f'_n(x_{n+1}) = \frac{q_n}{1 + x_{n+1} - \phi_n} = \frac{(1+\nu)^{n-1} q_1}{x_{n+1} - x_n + (q_1/b)(1+\nu)^{n-1}}. \hspace{1cm} (27)$$

Using (23), it is clear that

$$\lim_{n \to \infty} f'_n(x_{n+1}) = 0. \hspace{1cm} (28)$$

Therefore, the scenario considered eventually violates condition (S), no matter how large $b$ is chosen. It is instructive to understand why this is the case.

At a basic and obvious level, the reason why (S) does not hold must be blamed on the fact that the marginal productivity $f'_n(\cdot)$ for any given technology $\rho_n$ falls "too fast". This occurs despite the elasticity of each $f'_n(\cdot)$ being constant and equal to -1 for all $k \in \mathbb{R}^n$. Nevertheless, this constant elasticity with respect to "variable capital" must not conceal the fact that the elasticity with respect to total capital (i.e., the sum $k+\phi_n$ of variable and fixed capital) falls towards $-\infty$ as the scale of operation grows. Thus, in order to have such elasticity remain finite, the production functions $f_n(\cdot)$ must be "scaled by the the level of operation". This is precisely the content of assumption (A.3) below. For our particular example, a simple way of having Condition (S) hold would be to decrease appropriately the curvature of each production function $f_n(\cdot)$ beyond its corresponding tangency point $x_n$. (As an extreme example, we could postulate $f'_n(\cdot) = 0$, for $x_{n+1} \geq k+\phi_n \geq x_n$.) This transformation preserves fully the above computations for the sequences $\{\phi_n\}_{n \in \mathbb{N}}$ and $\{x_n\}_{n \in \mathbb{N}}$, but has Condition (S) satisfied thus making sustained growth possible at equilibrium. (Cf. Figure 3.)

25
Theorems 2 and 3 provide different necessary and sufficient conditions for sustained growth. Ideally, we would like to reach (or at least approximate) a situation where a single condition acts as both necessary and sufficient. We shall be able to do this below, but only for the special case where \( p = 1 \) (i.e., when there is no friction) and the technological externalities are not too acute.

The following additional assumptions will be required.

(A.3) Let \( g_n(z) = f_n'(z, \phi_n) \) and \( Z_n = \{ z \in \mathbb{R}_+: \phi_n \leq z \leq (g_n)^{-1}(\beta) \} \). The functions \( \{g_n(\cdot)\}_{n \in \mathbb{N}} \) have a uniformly bounded (negative) elasticity in their respective sets \( \{Z_n\}_{n \in \mathbb{N}} \).

(A.4) Let \( Q = \{ z \in \mathbb{R}_+: \hat{\psi}(z) = \psi(z) \} \). \( \exists \zeta > 0: \forall z \in \mathbb{R}_+, \exists z' \in Q, z' > z, \) such that \( z' \leq \zeta z \).

(A.5) Let \( \mathcal{F} = \{ \rho_n \in \mathcal{F} \text{ s.t. } \exists z \in \mathbb{R}_+: f_n(z, \phi_n) = \hat{\psi}(z) \} \). For all \( \rho_n \in \mathcal{F} \), \( \mathcal{F}(\rho_n) \cap \{ \rho_{n'} \in \mathcal{F}: n' > n \} = \emptyset \).
Assumption (A.3) is a technical condition, partly motivated by our discussion of the previous example. It requires that the "curvature" of any given production function reflects its "scale of operation", at least in the range where its marginal productivity is no lower than the critical value $\beta$. On the other hand, (A.4) simply assumes that, along the upper frontier of the set $\mathcal{F}$, the rate at which different technologies increase their respective scale of operation is uniformly bounded. Finally, (A.5) postulates that every technology $\rho_n$ which contributes to the global technological envelope of the economy must provide access (through its set $\mathcal{F}(\rho_n)$) to some other more sophisticated technology also belonging to such envelope.

As advanced, we want to focus on an scenario where technological externalities are not too intense. To express formally this idea, we shall say that technological externalities are of scale $\nu \in (0,1]$ if

$$\forall n \in \mathbb{N}, \sum_{s=n}^{\infty} \mu(\rho_s) \geq \nu \Rightarrow F_n(\mu,k) = f_n(k). \quad (29)$$

Which leads us to state our final result of this subsection.

**Theorem 4:** Assume (A.1) to (A.5), and $p = 1$.

(a) If there is an equilibrium $(\sigma, \chi)$ whose consumption path $(c(t))_{t=0}^{\infty}$ satisfies $\lim_{t \to \infty} c(t) = \infty$, then $\lim_{k \to \infty} \hat{\psi}'(k) \geq \beta$.

(b) If $\lim_{k \to \infty} \hat{\psi}'(k) > \beta$, $\exists \bar{\nu} > 0$ such that, provided technological externalities are of scale $\nu \leq \bar{\nu}$, then there is an equilibrium $(\sigma, \chi)$ whose consumption path $(c(t))_{t=0}^{\infty}$ satisfies $\lim_{t \to \infty} c(t) = \infty$.

**Proof:** See the Appendix.

Part (a) of the preceding result states a necessary condition for sustained growth which is a straightforward corollary of Theorem 2 for a context with no friction. On the other hand, its Part (b) establishes that such condition is also "nearly" sufficient, provided technologies are of sufficiently small scale. Thus, in a context with no friction, we will "almost
always" be able to ascertain whether or not unbounded growth is possible at some equilibrium of the economy. (10)

Reminiscent conditions have been discussed in the literature in relation with the issue of sustained growth (see, for example, Jones & Manuelli (1990)). However, there are at least two significant differences with this literature.

First, even though unbounded growth will always exist as a possibility if the sufficient condition is met, such indefinite growth is never guaranteed (cf. Proposition 1).

Secondly, no bounds on the extent to which limit marginal returns may exceed the discount rate are needed (as they indeed are in the received literature) in order to make sure that a well-defined equilibrium exists. Indeed, it would be perfectly possible to accommodate for global (and unbounded) increasing returns in our context and still ensure the existence of equilibrium. However, in order not to complicate matters in ways that are peripheral to our main concerns, we have indirectly abstracted from these issues through assumption (A.4). (Note that this assumption implicitly bounds through \( \zeta \) the range of increasing returns.)

3.3. Optimality, Expectations, and Equilibrium Selection

Thus far, we have talked only about the possibility of sustained growth. For, in view of Proposition 1, there is always some equilibrium of the model in which the growth of the economy eventually comes to a halt. In view of this state of affairs, an important question to ask is whether some sort of institutions may remedy this indeterminacy, narrowing down the set of possible outcomes to those which are desirable in some well-defined sense.

Two questions need to be settled:

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10 What occurs at the knife-edge case where the weak inequality of (a) holds with equality boils down to questions of rates of convergence, which are of little theoretical interest.
(i) What institutions are to be considered;
(ii) What is to be judged a desirable outcome.

We first address (ii). In this respect, a natural approach is to identify the concept of desirable outcomes with those that solve the traditional "planner’s problem". That is, the problem of maximizing the discounted utility of the (representative) consumer subject to the feasibility constraints of the economy. Formally, this problem is formulated as follows:

$$\max_{t=1}^{\infty} \delta^t U(c(t))$$

with respect to the sequences (11)

$$\{c(t), S(t), K(t)\}_{t=1}^{\infty};$$  \hspace{1cm} (30)

$$\{\lambda(t)\}_{t=1}^{\infty}, \lambda(t): \mathcal{T} \rightarrow \mathbb{R}_+;$$  \hspace{1cm} (31)

$$\{\gamma(t)\}_{t=1}^{\infty}, \gamma(t): \mathcal{T} \rightarrow \mathcal{B}(\mathbb{R}_+);$$  \hspace{1cm} (32)

and subject to:

$$c(t) + S(t) \leq X(t),$$  \hspace{1cm} (33)

$$\text{supp} \lambda(t) \subseteq \mathcal{T}(\tilde{\lambda}(t)),$$  \hspace{1cm} (34)

$$\mu(t) = p \lambda(t) + (1-p) \mu(t-1),$$  \hspace{1cm} (35)

$$K(t) = \sum_{\rho_n \in \mathcal{T}} \left\{ \int k \, d\left(\gamma(t)(\rho_n)\right)(k) \right\} \mu(t)(\rho_n) \leq S(t-1),$$  \hspace{1cm} (36)

$$X(t) = \sum_{\rho_n \in \mathcal{T}} \left\{ \int F_n(\mu(t), k) \, d\left(\gamma(t)(\rho_n)\right)(k) \right\} \mu(t)(\rho_n),^{(12)}$$  \hspace{1cm} (37)

11 Each $\lambda(t) \in \Delta(\mathcal{T})$ is to be interpreted as the technological profile of the firms which can revise their technology at $t$. On the other hand, each $\tilde{\lambda}(t)$ indicates the profile of capital use among the firms using each technology. (Of course, that associated to those technologies not currently in use is irrelevant.)

12 Here we incur in an obvious abuse of notation by associating each density $\mu(t)(\rho_n)$ to the corresponding production function $F_n(\cdot)$. 

29
where \( \mu(0) \) and \( S(0) \) are given. The interpretation of the decision variables and constraints in the above optimization problem is as follows. In (31), each \( \lambda(t) (p_n) \) represents the fraction of firms that adopt technology \( p_n \) among those which can revise their technology at \( t \) (cf. the interpretation of (35) below). On the other hand, every \( \gamma(t)(p_n) \) in (32) stands for the distribution of capital use among those firms adopting technology \( p_n \) at \( t \). Constraints (33), (36) and (37) are self-explanatory. As for the other two constraints, (34) simply specifies that the planner can only make firms choose a technology that is available given the current technological frontier. Finally, (35) requires that any adjustment of the economy-wide technological profile can only be performed through the fraction \( p \) of firms which currently enjoy the opportunity of technological revision. Thus, in this respect, the planner is assumed constrained by the firms’ own technological inertia.

For short, any path of variables which belongs to (or is induced by) a solution \( \{c^*(t), S^*(t), K^*(t), \lambda^*(t), \gamma^*(t) \}_{t=1}^{\infty} \) to the planner’s problem will simply be called optimal.

Once settled (ii), the first natural question to ask in connection with (i) is whether there is some market equilibrium (the basic "institution" defined in Section 2) which supports the solution to the planner’s problem. The answer to this question must be negative in general, as established by Proposition 2 below. (13)

**Proposition 2:** Assume (A.1)', (A.2) and (S). \( \exists \bar{p} > 0 \) such that if \( p \leq \bar{p} \), \( \exists \delta < 1 \) for which if \( \delta \geq \delta \) then there exist no equilibrium \((\sigma, \chi)\) whose consumption path is optimal. Moreover, for any such equilibrium path \( \{c(t)\}_{t=0}^{\infty}, \limsup_{t \to \infty} c(t) < \liminf_{t \to \infty} c^*(t) \), where \( \{c^*(t)\}_{t=0}^{\infty} \) is any optimal consumption path.

**Proof:** See the Appendix.

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13 Indeed, something much weaker than Condition (S) would be sufficient to yield the conclusion of the Proposition. (Essentially, only the existence of some feasible transition to a higher-index technology from the economy’s initial conditions would be enough.) We stay with Condition (S) in order to smoothen the presentation.
By Proposition 2, if optimality is to be ensured, the economy cannot rely solely on the market mechanism in some cases. In those circumstances where the market mechanism fails, an appropriate policy of subsidies could be devised that achieves the desired optimality (albeit perhaps at some implementation cost). For, as is customary in other models with externalities, the reason why the market may induce sub-optimal outcomes in our context must be blamed on its impossibility of internalizing some external effects of private decisions; in particular, those external effects on future technological adoption which are induced by such current decisions.

Although in principle possible, the task of finding the optimal subsidy policy which (in some well-defined sense) achieves the desired optimality is bound to be a very complicated problem. Instead, our focus here will be quite different and two-fold.

First, we shall identify circumstances where there is always some market equilibrium which is able to support optimal allocation paths. As it turns out, this will always occur when the economy is subject to no friction and the optimal path is "regular" (see Theorem 5 below).

Secondly, we shall concern ourselves with the following question of "equilibrium selection": What can a planner do (i.e., what "institutions" can she devise) in order to enforce the selection of the right (i.e., optimal) equilibrium, when several of them, some suboptimal, exist. We shall pose the question as a two-stage game between the planner and the population and realize that, at zero cost, the planner has a certain commitment strategy which induces the desired outcome.

The first of the above issues is addressed by the next result. For simplicity, it focuses directly on what will be called a regular optimal (consumption) path. This is the consumption component of an optimal path \( \{c^*(t), S^*(t), K^*(t), \lambda^*(t), \gamma^*(t)\}_{t=1}^{\infty} \) whose induced sequence \( \{\mu^*(t)\}_{t=0}^{\infty} \) of technological profiles (cf. (35)) satisfies:

\[
\forall \rho_n \in \text{supp} \ \mu^*(t), \ \mu^*(t)(\rho_n) > v,
\]

where \( v \) stands for the scale of the technological externalities, as defined by (30) above. It is easy to provide conditions on the underlying economic
environment which guarantee that any optimal path is regular, provided the scale of technological externalities is small enough. Rather than doing so, we shall assume such regularity from the start, in order to focus our attention on what are the main points of our present analysis.

**Theorem 5**: Assume \( p = 1 \). Every regular optimal path \( \{c^*(t)\}_{t=0}^{\infty} \) belongs to some equilibrium \((\sigma, \mathcal{X})\).

**Proof**: See the Appendix.

Theorem 5 establishes that every regular solution to the planner's problem is supportable by some equilibrium when there is no friction. This latter proviso \( (p = 1) \) cannot be dispensed with. For, not only the conclusion fails to be true when \( p \) is small enough (cf. Proposition 2) but one can also construct examples in which it fails for any \( p < 1 \). \(^{(14)}\) (It remains an open question, however, whether the long-run limit behavior differs significantly from the optimal one when \( p \) is sufficiently large; I conjecture that the answer should be negative.)

Given the wide multiplicity of equilibria inherent to our context, results like that above can just represent part of the story. For, in all cases where, for example, sustained growth is optimal, the possibility of supporting this outcome at equilibrium always coexists with the existence of other equilibria with bounded growth (recall Proposition 1). Therefore, a fully satisfactory approach to the problem should ensure, beyond the mere

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\(^{(14)}\) Since this point is not central to our concerns, I just sketch the idea underlying these examples. Suppose that the optimal solution demands that the economy moves steadily along the upper frontier of the global production possibility set \( \mathcal{X} \), which turns out to involve "long stretches" of points which are convex combinations involving different technologies. This sequence of "instantaneous situations" where, at the corresponding shadow prices, all firms make the same profits can only be supported at an equilibrium if \( p = 1 \). For lower values of \( p \), those firms adopting a less advanced technology (to which they may still be attached in the future with positive probability) should be receiving higher instantaneous profits if they are going to enjoy the same expected payoffs as the rest (something which is required at equilibrium if different firms are supposed to choose different technologies). But then, this requirement will shift the equilibrium away from the upper frontier, on which the optimal path was supposed to lie.
supportability at equilibrium of optimal outcomes, the existence of additional mechanisms (or "institutions") which could successfully pin down the desired equilibria.

A "planner" who has also tax/subsidy tools at its disposal will be the natural possibility considered here. In order to distinguish the somewhat fictitious role of the planner (whose purpose is mainly instrumental in defining an optimal path) from that more "institutional" which is now considered, we shall speak instead of a "government", whose policy instruments are taxes and subsidies, as described below. Specifically, we shall assume that the government and the population are involved in a two-stage game $\mathcal{G}$ in which:

(a) the government moves first, committing itself to a tax/subsidy scheme that will prevail thereafter;

(b) given the payoffs induced by the former scheme, the population plays an equilibrium as described above.

First, we need to specify in detail which type of tax/subsidy schemes we shall consider. Denote by $\Xi$ the set of allocations of the form $\chi = (c,S,\mu,K,\pi,r)$ which fully describe the global situation of the economy at any given point in time (cf. Definition 1). A tax/subsidy scheme (hereafter, referred to simply as a "scheme") is identified with a function

$$\varphi: \Xi \times \mathbb{N} \rightarrow \mathbb{R}^T$$

which specifies, for any current allocation $\chi(t) \in \Xi$ prevailing at time $t$ the tax or subsidy $\varphi_n(\chi(t),t)$ to which any firm adopting technology $\rho_n$ at $t$ is subject or entitled to at this time (negative values will be interpreted as taxes, positive values representing subsidies).

For simplicity, it will be assumed that the government must balance the budget every period.\(^{15}\) Thus, for all $t$ and $\chi \in \Xi$, it is required that:

\(^{15}\) This could be easily generalized to have the government confront an intertemporal budget constraint, even though this would bring in modelling problems which are not central to our concern here and are best avoided. For example, it would have to be decided whether the government should be modelled to confront prices (i.e., interest rates) parametrically or rather have it exploit monopolistic advantages.
\[ \sum_{\rho_n \in \mathcal{T}} \varphi_n(\chi,t) \mu(\rho_n) = 0, \quad (40) \]

where the technological profile \( \mu \) above is the one included in the contemplated \( \chi = (c,W,\mu,K,\pi,r) \). Furthermore, overall feasibility requires that aggregate tax should not exceed the current aggregate resources of the economy. That is,

\[ \frac{1}{2} \sum_{\rho_n \in \mathcal{T}} |\varphi_n(\chi,t)| \mu(\rho_n) \leq X(\chi), \quad (41) \]

where \( X(\chi) \) abuses the notation of (37) in the obvious fashion. The set of allowable schemes which satisfy (40) and (41) represents the strategy space of the government in the game \( \mathcal{G} \).

Preferences are postulated as follows. For the agents in the population, firms and consumers, as described above. (Now, the payoff of a firm using technology \( \rho_n \) at time \( t \) is given by its net profits resulting from adding to the gross profits \( \pi^*_{\mu}(\mu(t),r(t)) \) the tax/subsidy \( \varphi_n(\chi(t),t) \) prescribed by the prevailing scheme.) For the government they are represented by the functional

\[ \sum_{t=0}^{\infty} \delta^t V(c(t),\xi(t)) \quad (42) \]

where

\[ \xi(t) = \sum_{\rho_n \in \mathcal{T}} |\varphi_n(\chi,t)| \quad (43) \]

measures the size of the intervention and, therefore, its induced distortionary cost. Naturally, we assume that \( V(\cdot) \) is decreasing in this second argument, whereas in the case of no intervention \( V(\cdot,0) = U(\cdot) \) as introduced in (6) above.

The description of the game \( \mathcal{G} \) completed so far is enough to carry our main objectives. Even though it is still somewhat incomplete, it will remain so in order to save the reader some tedious notation.
The first issue to deal with is, of course, that of existence of equilibrium in the game $\mathcal{G}$. This is bound to be a complex question which, for our purposes, need not be confronted directly if we make the assumption that a solution to the planner's optimization problem exists, i.e., that there exists an optimal path. Sufficient conditions for this state of affairs were discussed in Remark 1 above. Under these circumstances, the following conclusion is obtained.

**Theorem 6:** Assume $p = 1$. If a regular optimal path exists, so does an equilibrium of $\mathcal{G}$, every one of which induces an optimal path. Moreover, this outcome is achieved at zero distortionary cost, i.e., $\xi(\cdot) = 0$ along the equilibrium.

**Proof:** See the Appendix.

The main idea underlying Theorem 6 can be easily outlined. The government, through the commitment power it enjoys as a first mover in the game $\mathcal{G}$, can make sure that no path different from the optimal one may survive as an equilibrium in the second stage. In this fashion, it can shape expectations towards the desired continuation equilibrium, i.e., the one supporting the optimal path, whose existence is guaranteed by Theorem 5. Moreover, once such commitment is in place, the fact that the induced path is an equilibrium outcome allows the government to leave all "threat" of intervention out of the equilibrium path. On the equilibrium path itself, no such intervention is necessary. Thus, being costly, it will be avoided at equilibrium.

**Remark 1:** Sequential moves, size, and commitment power.

Even though we have presented the government’s commitment power in terms of its first-mover advantage, this is essentially a modelling strategy. Simultaneous moves, or even moves in reverse order, would still yield identical conclusion although in a less natural model. (For example, the population could be modelled as choosing some equilibrium associated to the "subsequent" determination of the government’s scheme.) Indeed, the key feature which underlies the commitment power of the government is its size, relative to the assumed insignificance of the other agents (firms and
consumers). It is this asymmetry in sizes which gives the government the power to "shape expectations" and select its desired equilibrium.

Remark 2: Open-loop versus closed-loop schemes

An additional feature of the model which underlies the conclusion of Theorem 6 is the particular nature of the contemplated schemes. Specifically, it is crucial that these schemes be what we could label "closed loop", i.e., schemes that have their prescriptions depend on the current allocation of the economy. Otherwise (for example, if they were "open loop", depending on time alone), the feedback between current allocation and associated payoffs which underpins the logic of our conclusion would not hold any longer. Consequently, the implementation problem could not be solved in general, certainly not at zero distortionary cost.

4.- SUMMARY AND CONCLUSIONS

In this paper, a strategic model of growth has been proposed with the following main characteristics:

(i) Firms are involved in a joint process of accumulation and technological advance which is subject to crucial externalities;
(ii) Expectations play a key role in selecting among the wide multiplicity of drastically different dynamic paths generally consistent with equilibrium;
(iii) Some type of institutions (e.g., an interventionist government) may help close the inherent indeterminacy of the model by directing the expectations of the population towards some desired equilibrium.

The paper has explored the interaction between the above listed characteristics of the model, exploring their implications under alternative assumptions on the environment. More specifically, it has aimed at identifying technological conditions which are either necessary or/and sufficient for sustained growth.
Much needs to be done in order to enrich the scenario and conclusions of the model proposed here. Let me single out two of them which I think especially important.

One has to do with the incorporation of genuine technological innovation into the model. No doubt that this aspect of technological change also exhibits much of the externalities and complementarities which represent the core of our approach. Thus, it seems likely that some of the recent developments in Growth Theory which address explicitly this issue (cf. the Introduction) could be fruitfully integrated with some of the basic components of our model. The already discussed work of Young (1992) might be a fruitful place to start in this task.

A second dimension of the model which should also be further enriched is its institutional sphere. There are a number of important institutions in the real world other than central government whose important role in issues "equilibrium selection" should not be ignored by future developments. Social conflict, lobby groups, social norms and customs, political decision making in general represent obvious candidates in this respect (see, for example, Matthews (1986) or North (1991)). Even though we have recently witnessed some important theoretical contributions in this direction (cf. Boylan, Ledyard & McKelvey (1991), or Persson & Tabellini (1992)) further advances in this important but difficult area would seem one of the priorities in the research agenda of growth theorists and development economists.
APPENDIX

Proof of Theorem 1:

The key idea of the proof is to view the strategic context described in Section 2 as a generalization of an anonymous sequential game (ASG), the concept proposed by Jovanovic & Rosenthal (1988). To do so, we first need to specify the individuals’ states. Following the suggestion in the text, the following ones are adopted: for the typical (representative) consumer at t, the currently available savings $S(t-1)$; for a typical firm $j$, the tuple $(d_j(t-1), \theta_j(t), \tilde{p}(t))$ given by its previous technological choice $d_j(t-1) \in \mathcal{T}$, the indicator variable for technological adjustment $\theta_j(t) \in \{0,1\}$, and the common prevailing technological frontier $\tilde{p}(t)$. As in Jovanovic & Rosenthal (1988) (hereafter referred to as JR for short), the "social states" are simply defined to be probability distributions over individual states. Thus, a typical one consists of the vector of distributions $\omega(t) = \left[ S(t-1), \mu(t-1), ((1-p), p), \tilde{p}(t) \right]$, where, for notational simplicity, $S(t-1)$ and $\tilde{p}(t)$ stand for the distributions concentrated on these points, the vector $((1-p), p)$ represents the proportions of firms which do, and do not, have the opportunity of revising its previous technological choice every period, and $\mu(t-1)$ is the preceding technological profile, as defined in (4). Note that, here, we are implicitly restricting to states where all (identical) consumers behave symmetrically. These will be the only ones considered. Thus, incorporating this feature into the framework from the start, we shall economize on some notation.

To complete the formulation of the game, payoff and transition functions need to be specified. The latter are simply determined by (9) and (15), given the agents’ strategies. As for the payoff functions, different issues arise in connection to firms or consumers.

To define firms’ instantaneous payoff functions we shall find it useful to define the function

$$\mathbb{P}: \Delta(\mathcal{F}) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \quad (45)$$
which associates to each technological profile $\mu$ and available savings $S$, a corresponding market-clearing interest rate $r(\mu, S)$. That is, an interest rate such that:

(a) For every technology $\rho_n$ and any given firm adopting it, the amount of capital selected is governed by some probability distribution $v_n(\mu, S) \in \mathcal{B}(\mathcal{R}_n)$ that satisfies:

$$\text{supp } v_n(\mu, S) \subseteq k_n(\mu, r(\mu, S)).$$

(b) Given those firms' investment decisions, the capital market clears, i.e.,

$$S = \sum_{\rho_n \in \mathcal{F}} \left\{ \int k \, dv_n(k) \right\} \mu(\rho_n).$$

Suppose that we restrict our attention to firms' strategies $\{\sigma^i_t\}_{t=1,2,\ldots}$ that, conditional on every given technological choice $\rho_n$, induce probability distributions on the first capital component $\{\sigma^i_t|\rho_n\}_{t=\rho_n}$ which satisfy (when well-defined):

$$\sigma^i_t|\rho_n = v_n(\mu(t), S(t-1)).$$

Then, we may concentrate on the technological choice of the firms as the sole strategic variable of the game. That is, firms may be viewed as selecting only their technological strategy, which is identified with the the marginal distribution in $\mathcal{B}(\mathcal{F})$ induced by any corresponding $\sigma^i_t$. An equilibrium found in these strategies can then be completed as required by (46) and (47) in order to construct a fully specified equilibrium of the game.

Two issues need to be confirmed in this respect. First, that the function $r(\cdot)$ introduced in (45) is well-defined (i.e., a market-clearing rate exists associated to every pair $(\mu, S)$). Second, that such function can be chosen continuous. Both of these requirements are immediate consequences of our assumptions. This implies that the payoff function for a firm which adopts technology $\rho_n$ at any time $t$ can be written as a continuous function $\pi_n(S(t-1), \mu(t-1), \sigma^i_t)$ of the current social state and firms' strategy (where
$\Delta(\mathcal{F})$ and $\mathcal{B}(\Gamma^1)$ are both endowed with the weak topology). This function is given by:

$$
\pi_n^*\left(S(t-1),\mu(t-1),\sigma_t^1\right) = \pi_n^*\left[\mathcal{M}(\mu(t-1),\sigma_t^1), \mathcal{I}(\mu(t-1),S(t-1))\right],
$$

(49)

where $\pi_n^*(\cdot)$ and $\mathcal{M}(\cdot)$ are defined in the text (cf. (15) and the paragraph following (12)).

The consumers’ instantaneous payoff function is given by $U(\cdot)$, a function of current consumption, as defined by (6). If we treat all consumers symmetrically through the representative-consumer construct, it is clear that the choice set contemplated in Subsection 2.4 ($\Gamma(\omega_c) = \mathbb{R}_+, \forall \omega_c \in \Omega_c$) is not applicable. In particular, the consumption $c(t)$ of the "representative consumer" at $t$ must be bound by the current availability of resources at $t$ given by:

$$
X(t) = S(t-1) \left(1+r(t)\right) + \pi(t).
$$

(50)

This poses some formal problems if we want to keep the format of an anonymous game, in the sense of JR. (This formulation requires that the choice set of an agent only depends on current individual or social states.) To remedy this problem, we simply replace the above function $U(\cdot)$ by a function $\bar{U}(S(t-1),\mu(t-1),c(t),\sigma_t^1)$ defined by:

$$
\bar{U}(S(t-1),\mu(t-1),c(t),\sigma_t^1) = \min \left\{ U(c(t)), U(X(t)) \right\}.
$$

(51)

Note that $X(t)$, as given by (50), can be conceived to be a function of $(S(t-1),\mu(t-1),\sigma_t^1)$, along the lines described above. (If useful, one may think of (51) as the reflecting a uniform rationing mechanism to be applied if consumers wanted to consume more than currently available.)

Given the instantaneous payoff functions just defined, our context is very similar to that of an anonymous game. Only two remaining differences with JR’s framework need to be tackled in order to adapt their existence result.

(i) The discount rate used by firms in evaluating their intertemporal strategy at some $i$, $(R(i,t))^{-1}$, is endogenous to the model since it depends on the prevailing rates of interest.
(ii) Payoff functions have not been assumed uniformly bounded.

Both issues can be addressed through a similar argument. Consider a context artificially restricted in the following two respects:

(a) Firms can only use technologies that are present at the start of the process (i.e., \text{supp} \mu(t) \subseteq \text{supp} \mu(0)).

(b) Aggregate capital (and thus savings) is bounded by some given \bar{K}, finite.

First, we shall show that an equilibrium exists for this artificial economy. Constraints (a) and (b) can be formally incorporated into the ASG framework in a straightforward fashion. As for (a), simply replace the general technology set \mathcal{T} by \mathcal{T}_o which includes only the technologies in \text{supp} \mu(0). With respect to (b), instead of (9) consider the following (continuous) law of motion for savings:

\[
S(t) = \max \{\bar{K}, S(t-1) \left(1 + r(t)\right) + \pi(t) - c(t)\}. \tag{52}
\]

In such modified context, existence of equilibrium follows from an adaptation of JR's existence result. On the one hand, (ii) is obviously satisfied. On the other hand, even though the firms' "discount rate" at each t, \( (1+r(t))^{-1} \), is endogenous (i.e., part of the equilibrium) it can be ensured to remain bounded below one for all feasible paths (not only along an equilibrium). This is enough to adapt JR's Lemma and guarantee the existence of a value function (and therefore, a well-defined solution) for the firms' decision problem. The counterpart for consumers follows directly from this Lemma.

I finally argue that any equilibrium found under restrictions (a) and (b) is also an equilibrium for the original economy, provided that \bar{K} is chosen large enough.

First, note that if an equilibrium is found under restriction (a), (2c) assures that it is also an equilibrium for the unrestricted context. (Specifically, no firm will ever want to adopt a technology unless it is chosen by a positive frequency of firms.)
Secondly, (A.1) implies that, if \( \bar{K} \) is large enough, every path having the corresponding constraint in (52) be binding at some point in time must, for an arbitrarily long time span, exhibit a marginal productivity of capital (and, therefore, market-clearing interest rate) below \((1-\delta)/\delta\). Obviously, this cannot be part of an equilibrium path. (See the proof of Theorem 2 below, for an elaboration on this claim in a slightly different, somewhat more general, context.) This implies that, as desired, every equilibrium found for the artificial economy can be ensured an equilibrium for the original context.

**Proof of Proposition 1:**

The first claim of the Proposition is a direct consequence of the fact that, as explained in the proof above, there is always an equilibrium satisfying \( \text{supp} \ \mu(t) \leq \text{supp} \ \mu(0) \). By (A.1) and (A.2), this implies that growth along any such equilibrium must be bounded.

The second claim follows from the following two considerations.

On the one hand, if \( p = 0 \), it is a strictly dominated response for every firm which could receive the option of revising its strategy at any \( t \) (even though this occurs only with probability zero) to adopt some technology \( \rho_n \notin \text{supp} \ \mu(t) \).

On the other hand, part (i) of (11) and (A.1)' imply that all future profits associated to any feasible technological choice adopted at \( t \) are bounded by the maximum \( M_n \) corresponding to the set of technologies \( \rho_n \in \mathcal{S}(\hat{\rho}(t)) \).

Thus, combining both facts, we conclude that there must exist some \( \bar{p} > 0 \) such that, if \( p \leq \bar{p} \), the gains for any firm of shifting at \( t \) to a technology \( \rho_n \notin \text{supp} \ \mu(t) \) may only start to be significant (but bounded) too far into the future (cf. (13)) to render such a decision worthwhile.
Proof of Theorem 2:

Let
\[ \lim_{k \to \infty} \hat{\psi}'(k) < \beta = \frac{1-\delta}{\delta} \]  
and assume, for the sake of contradiction, that the conclusion of the theorem is violated, i.e., there is unbounded growth. For simplicity, we shall focus on the case with \( p = 1 \), the argument then readily extendable to a context with sufficiently large \( p < 1 \). Along any equilibrium of the process, the following FONC must be satisfied, at all \( t = 1, 2, \ldots \), for the representative consumer’s decision problem \( P_c(t) \):

\[ \frac{U'(c(t))}{U'(c(t+1))} = \delta \left( 1+r(t+1) \right). \]  

At some \( t_0 \), (53) and profit maximization by firms imply that we must have

\[ r(t_0) < \beta. \]  

Indeed, if unbounded growth is to prevail, such inequality must hold also for all future \( t \) for which \( K(t) \geq K(t_0) \). But, of course, such inequality can not be maintained continuously along an equilibrium path with indefinite growth. If it did, (54) would imply that while the capital stock of the economy grows indefinitely, consumption is continuously decreasing. (Recall the assumed concavity of \( U(\cdot) \), which implies non-increasing first derivative.)

Thus, given any \( M \) arbitrarily large, let \( t_1 \) and \( t_0 \) \( (t_1 > t_0) \) be chosen such that:

(a) \( K(t_1) - K(t_0) \geq M; \)
(b) \( r(t) < \beta \), for all \( t = t_0, t_0+1, \ldots, t_1; \)
(c) \( r(t_1+1) \geq \beta. \)

The assumption that growth is unbounded, together with (53), implies that such \( t_0 \) and \( t_1 \) can always be found. By (c) and firms’ profit maximization, we must have \( K(t_1+1) \leq K(t_0) \). Thus, by (a), \( c(t_1) \geq M. \) On the other hand, by (b) and (54), \( c(t_1-1) < c(t_0) \). Combining these inequalities, we have:
\[(1+r(t_1)) \delta = \frac{U'(c(t_1-1))}{U'(c(t_1))} = \frac{U'(c(t_0))}{U'(M)} \quad (56)\]

Since \(c(t_0)\) is bounded above independently of \(M\), and \(M\) itself can be chosen arbitrarily large, \(r(t_1)\) can also be forced arbitrarily large. (Recall that it was assumed that \(\lim_{c \to \infty} U'(c) = 0\).) This would render the situation at \(t_1\) incompatible with firms' profit maximization.

**Proof of Theorem 3:**

The proof is decomposed into three steps.

(i) First, I consider a context with any finite number of available technologies and show that, under condition (S) and \(p = 1\), growth takes place until the growth possibilities allowed by the "more advanced" technology are fully exhausted.

(ii) Second, I show that the previous case can be "extended" to an infinite number of technologies, preserving existence of equilibrium, and inducing sustained growth.

(iii) Third, I argue that the aforementioned conclusions also hold for large enough \(p < 1\), if the above mentioned equilibrium is modified appropriately.

Assume, for the moment, that \(p = 1\), and suppose that only some finite number of technologies \(\{r_n\}_{n=0,1,\ldots,N}\) are available (recall that \(r_0\) stands for "inaction"). For each of these technologies, let

\[\bar{k}_n = \min \{k \in \mathbb{R}_+: f'_n(\bar{k}_n) - \beta = \eta_1\}, \quad (57)\]

where \(\eta_1\) is as contemplated in condition (S). For simplicity, further assume that, for each \(n = 1,2,\ldots,N-1\), \(n+1\) may play the role of \(n\) in Condition (S).

Based on the capital thresholds defined in (57), we now proceed to construct an equilibrium which satisfies:

\[\lim_{t \to \infty} f'_N(K(t)) = \beta. \quad (58a)\]
\[ \lim_{t \to \infty} \mu(t) = e_n, \] (58b)

where \( e_n \) denotes the technological profile in \( \Delta(\mathcal{F}) \) which is concentrated in technology \( \rho_n \).

First, we construct an artificial economy where an equilibrium of the desired type exists. Then, we show that it is also an equilibrium for the original economy. Consider an economy where the choice set of firms is dependent on the social state \( \omega \in \Omega \) and satisfies:

\[ \Gamma^f(\omega) = \mathbb{R}_+ \times \{ \rho_n \}, \text{ for } \bar{k}_n > S(t-1) > \frac{\bar{k}_n}{n+1}; \] (59a)

\[ \Gamma^f(\omega) = \mathbb{R}_+ \times \{ \rho_n \rho_{n+1} \}, \text{ for } S(t-1) = \frac{\bar{k}_n}{n}; \] (59b)

where, for notational convenience, we denote \( \bar{k}_0 = 0 \) and \( \bar{k}_N = +\infty \). Clearly, \( \Gamma^f(\cdot) \) defines a continuous correspondence on \( \Omega \). (Note that if \( p = 1 \) the set of social states can be simply identified with \( \mathbb{R}_+ \), the set of possible savings.) Thus, we can adapt the existence result of JR along the lines of the proof of Theorem 1 to guarantee the existence of an equilibrium \((\bar{x}, \bar{y})\).

I claim now that this equilibrium can be chosen to exhibit the desired performance. On the one hand, note that for all \( t \) at which current (equilibrium) savings \( \bar{S}(t-1) \) satisfy:

\[ \bar{S}(t-1) \leq \bar{k}_{N-1}, \] (60)

the associated equilibrium interest rate must fulfill:

\[ \bar{r}(t) > \beta. \] (61)

Therefore, by the the strict concavity of \( U(\cdot) \) and the first-order conditions (54) prevailing along any equilibrium, there must be some \( t_0 \) such that, if \( t \geq t_0 \), then:

\[ \bar{\mu}(t) = e_n; \quad \bar{S}(t-1) \geq \bar{k}_{N-1}. \] (62)

This implies that every equilibrium of the artificial economy induced by (59) exhibits the desired performance. The conclusion that any such equilibrium is also an equilibrium of the original economy (where technological choices of
firms are unrestricted) is a direct consequence of condition (S). This completes Part (i) of the proof.

To address Part (ii), let \( (\mathcal{O}_N, \mathcal{X}_N) \) be an equilibrium satisfying (58) for the economy whose technologies are restricted to lie in the set \( \{ \rho_n \} \). By (2c), such equilibrium is also an equilibrium of the original economy with the full range \( \mathcal{S} \) of technologies. Consider the sequence \( \{(\mathcal{O}_N, \mathcal{X}_N)\}_{n=1}^{\infty} \). Every \( (\mathcal{O}_N, \mathcal{X}_N) \) can be appropriately identified as an element of a Cartesian product of compact sets indexed by \( t = 1, 2, \ldots \). (Note that the set feasible actions at any given \( t \) can be constrained to lie in an appropriately large bounded subset of an Euclidean space.) Thus, by Tychonoff's Theorem, such Cartesian product is compact, with the product topology. Consequently, some subsequence of \( \{(\mathcal{O}_N, \mathcal{X}_N)\}_{n=1}^{\infty} \) has a well-defined limit, say \( (\mathcal{O}^*, \mathcal{X}^*) \).

The strategy profile \( (\mathcal{O}^*, \mathcal{X}^*) \) induces unbounded growth. However, it still needs to be verified that it defines an equilibrium of the process. Since every \( (\mathcal{O}_N, \mathcal{X}_N) \) is itself an equilibrium, this boils down to confirming that the agents' payoff functions are continuous at \( (\mathcal{O}^*, \mathcal{X}^*) \). Since the issues involved are straightforward, let me simply sketch the argument. The crucial point has to do with the behavior of the payoff functions "at infinity", i.e., at the tail of the sequences induced by these strategies. Since agents discount payoffs at a rate bounded above zero, an obvious way of guaranteeing the desired continuity is by ensuring that the agents' payoffs do not grow too fast. This can be done, for example, by making the choice sets in (59) dependent not only on the state but also on time. Thus, suppose we contemplate a sequence of times \( \{ t_0(=0), t_1, t_2, \ldots \} \) and postulate:

\[
\Gamma^f_t(\omega) = \mathbb{R}_+ \times \{ \rho_n \}, \begin{cases} 
\text{for } S(t-1) > \bar{\kappa}_{n-1} & \text{and } t_n > t \geq t_{n-1} ; \\
\text{for } \bar{\kappa}_n > S(t-1) > \bar{\kappa}_{n-1} & \text{and } t \geq t_{n-1} ; 
\end{cases} \tag{63a}
\]

\[
\Gamma^f_t(\omega) = \mathbb{R}_+ \times \{ \rho_n \rho_{n+1} \}, \text{ for } S(t-1) = \bar{\kappa}_n & \text{and } t \geq t_n ; \tag{63b}
\]

This is a generalization of (59) which, for every \( t \), still guarantees the continuity of the corresponding choice correspondence \( \Gamma^f_t(\cdot) \). Thus, relying again on the existence result of JR, we can assure the existence of equilibrium in this case for any finite number \( N \) of considered technologies.
But now, making the time span \((t_n-t_{n+1})\) sufficiently long for each \(n = 1,2,\ldots,N\) the rate of growth of the economy can be appropriately controlled, as required in order to ensure the continuity of the payoff functions at infinity as \(N \to \infty\).

The proof is now complete for the case \(p = 1\). To extend the argument for sufficiently large \(p < 1\) (Part (iii) above), one needs to modify the above approach by contemplating choice-set correspondences for firms of the form \(\tilde{f}_i^t(\omega, d, \theta)\), which depend not only on the social state \(\omega\), but also on each firm’s previous technological choice \(d\) and its particular realization of the variable \(\theta \in \{0,1\}\). To verify that the above argument applies to this case also, as long as \(p\) is sufficiently close to one, the key point to note is the following: the bounds \(\eta_1, \eta_2, \eta_3 > 0\) contemplated in Condition (S) can be combined with the assumed uniform bound on first and second derivatives of the functions \(F_n(\cdot)\) to establish an equally uniform approximation to the case \(p = 1\) through a context with sufficiently small friction.

**Proof of Theorem 4:**

Part (a) is simply a re-statement of Theorem 2. As for Part (b) of the theorem, we shall rely on an argument quite inspired in that used in the previous proof, applied again to an artificial and economy. In the present case, however, such artificial economy will involve not only a re-definition of the choice-set correspondences but also of the production functions.

Assume, for simplicity, that every technology \(p_n, n = 1,2,\ldots\), "contributes" some point to the upper-envelope function \(\tilde{\psi}(\cdot)\). That is

\[
\forall p_n \in \mathcal{T}, \exists z \in \mathbb{R}_+: f_n(z; \phi_{p_n}) = \tilde{\psi}(z).
\]  

(64)

(Otherwise, we may simply ignore those technologies that do not.) For each \(n = 1,2,\ldots\), denote:

\[
\mathcal{Y}_n = \bigcup_{n' = 1}^{n} Y_{n'},
\]

(65)
i.e., the production possibility set induced by the first \(n\) technologies. Associated to each \(\mathcal{Y}_n\), define a "fictitious" production function \(h_n(\cdot)\): \(\mathbb{R}_+ \to \mathbb{R}_+\) by:

48
\[ h_n(z) = \max \{ x \in R_+ : (x, z) \in \mathcal{J}_n \}, \]  
\[ h_n(z) = \max \{ x \in R_+ : (x, z) \in \mathcal{J}_n \}, \]  
(66)

where \( \mathcal{J}_n \) denotes the convex hull of \( \mathcal{J}_n \). Note that, by (A.4), every \( h_n(\cdot) \) is well-defined since \( \hat{\psi}(\cdot) \) is also assumed so. In order to exploit useful analogies with the proof of Theorem 3, let \( \mathcal{D} = \{ \Theta_n \}_{n=1}^{\infty} \) stand for the set of technologies respectively induced by the production functions \( \{ h_n \}_{n=1}^{\infty} \). Each of these technologies are to be conceived as displaying zero fixed costs. (Thus, in particular, higher-index technologies weakly dominate lower-index ones at all scales of production; see Figure 4 below.)

For each \( \Theta_n \in \mathcal{D} \), let

\[ \hat{z}_n = \min \{ z \in R_+: h_n(z) = \hat{\psi}(z) \} \]  
(67a)

\[ \hat{\hat{z}}_n = \max \{ z \in R_+: h_n(z) = \hat{\psi}(z) \} \]  
(67b)

By construction (cf. (65) and (66)), we obviously have \( \hat{\hat{z}}_n \leq z_{n+1} \) for all \( n = 1, 2, \ldots \). On the other hand, by (A.4), we must also have that

\[ \zeta \hat{\hat{z}}_n \geq z_{n+1}. \]  
(68)

Furthermore, under the assumption that \( \lim_{k \to \infty} \hat{\psi}'(k) > \beta \), there is some \( \alpha \) such that:

\[ \forall n = 1, 2, \ldots, \hat{\psi}'(z_n) = h_n'(z_n) > \alpha > \beta. \]  
(69)

Together with the sequences \( \{ z_n \}_{n=1}^{\infty} \) and \( \{ \hat{z}_n \}_{n=1}^{\infty} \) our argument will use (as a counterpart here of the capital thresholds defined in (57)) the sequence \( \{ \hat{z}_n \}_{n=1}^{\infty} \) defined, \( \forall n = 1, 2, \ldots \), by:

\[ \hat{z}_n = \min \{ z \in R_+: h_n'(z) = \alpha \}. \]  
(70)

By (A.1), every \( \hat{z}_n \) is well defined.
Consider now an artificial economy such as the one considered in the proof of Theorem 3, with the production functions \( \{h_n\}_{n=1}^{\infty} \) defined in (66), and time-varying choice-set correspondences

\[
\Gamma_t^f(\omega) = \mathbb{R}_+ \times \{\omega_n\}, \quad \begin{cases} 
\text{for } S(t-1) > z_{t-1} & \& t_t > t \geq t_{t-1} ; \\
\text{for } z_t > S(t-1) > z_{t-1} & \& t \geq t_{t-1} ; \\
\text{for } S(t-1) = z_t & \& t \geq t_t ;
\end{cases} \quad (71a)
\]

\[
\Gamma_t^f(\omega) = \mathbb{R}_+ \times \{\omega_n, \omega_{n+1}\}, \quad \text{for } S(t-1) = z_t & \& t \geq t_t ; \quad (71b)
\]

for some pre-established sequence of times \( \{t_1, t_2, \ldots\} \). (Here, we make \( t_0 = z_0 = 0 \).) Further assume that, unlike for what has been postulated for the model in the text, our "artificial" economy (based on choice set correspondence (71) and "fictitious" production functions \( \{h_n\}_{n=1}^{\infty} \)) exhibits no externalities of the sort contemplated in (2). (Thus, any firm can enjoy the full potential of the technology \( \omega_n \) that it uses, according to the corresponding function \( h_n(\cdot) \).) The existence of an equilibrium for such economy (as in Definition 1) follows from our earlier arguments adapted to the present context. (First, we would truncate the context to a finite number of technologies; then, allow this number to expand slowly enough.)

As before, our next objective is to argue that such an equilibrium found for the artificial economy is also applicable, at least when technological externalities are of sufficiently small scale, to the original economy (i.e., to the context with the original technological possibilities given by \( \{\rho_n\}_{n=1}^{\infty} \) and constant choice-set correspondences). To establish this claim, first notice that, given (A.3), there must exist some \( H > 0 \) such that:

\[
\forall n = 1, 2, \ldots, \quad \frac{z_n}{z_{n+1}^2} > H. \quad (72)
\]

Thus, suppose that technological externalities are of scale \( v \leq H \). Then, along the equilibrium path, all prevailing points on the corresponding "fictitious" production function can be appropriately interpreted as points achievable by firms which confront parametrically the corresponding interest rate and fully achieve the productive potential of the technology that they are currently using. In other words, the assumed lack of externality effects is indeed appropriate along the equilibrium path, even if we conceive it as a path
realized within the original economy (see Figure 4). On the other hand, by essentially the same argument applied in the proof of Theorem 3, the other constraints contemplated by our artificial economy (in particular (71)) are also seen to be non-binding along the equilibrium path, even within the original context. This completes the proof.

[see Figure 4 on pag. 50 above]

Proof of Proposition 2:

As in Proposition 1, we can assert that \( \exists \bar{p} > 0 \) such that if \( p \leq \bar{p} \) then every equilibrium must induce bounded growth. Specifically, depending on the initial conditions, there exists some finite \( M_0 \) such that

\[
\limsup_{t \to \infty} c(t) \leq M_0
\]

(73)

for any consumption path \( \{c(t)\}_{t=0}^{\infty} \) belonging to some equilibrium. The crucial point to note is that this conclusion is independent of \( \delta \), the discount rate of the planner (and consumers). Thus, if some feasible path \( \{\xi(t)\}_{t=0}^{\infty} \) exists such that

\[
\liminf_{t \to \infty} \xi(t) \equiv M_1 > M_0,
\]

(74)

the former path cannot be optimal for some sufficiently large discount rate. Indeed, the fact that such latter path exists is a consequence of Condition (S).

Proof of Theorem 5:

The key point to observe is that, along every regular optimal path \( \{c^*(t), S^*(t), K^*(t), \lambda^*(t), \phi^*(t)\}_{t=1}^{\infty} \) with induced sequence \( \{\mu^*(t)\}_{t=0}^{\infty} \) of technological profiles, the following conditions must be satisfied at all \( t = 1, 2, \ldots \):

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\( \forall \rho_n \in F_n, \quad \forall k \in \text{supp} \phi(t)(\rho_n), \quad \mu^*(t)(\rho_n) > 0, \quad k \in \text{supp} \phi(t)(\rho_n) \Rightarrow \mu^*(t)(\rho_n') > 0, \quad k' \in \text{supp} \phi(t)(\rho_n') \)

(a) \( \frac{\partial F_n}{\partial k} \left( \mu^*(t), k \right) = \frac{\partial F_n}{\partial k} \left( \mu^*(t), k' \right) = \nu^*(t) \)

(b) \( F_n(\mu^*(t), k) - \nu^*(t)(k+\phi_n) = F_n(\mu^*(t), k') - \nu^*(t)(k+\phi_n') \approx \pi^*(t) \)

\( \forall \rho_n \in F_n(\rho(t)), \forall k \in \mathbb{R}, F_n(\mu^*(t), k) - \nu^*(t)(k+\phi_n) \approx \pi^*(t). \)

The basis for this claim derives from straightforward considerations of production efficiency. In other words, if (75) and (76) did not hold at some \( t \), the planner could slightly reallocate resources across firms and available technologies at \( t \) and produce more output with the same amount of capital \( K(t) \). Since such reallocation would not alter either the production potential of the technologies in use (recall (38)), nor affect the set of technologies available next period, \( F_n(\rho(t+1)) \), this would contradict the supposed optimality of the contemplated path.

Once confirmed that (75) and (76) must hold every period, a standard argument on "shadow prices" allows us to assert that the optimal path can be decentralized through prices, i.e., interest rates, \( \{\nu^*(t)\}_{t=0}^\infty \), as these are defined in Part (a) of (75). For firms this follows from (76) since \( p = 1 \) (thus, firms take their decisions on the basis of instantaneous profits alone). As for consumers, the conclusion obtains from the fact that \( \{\nu^*(t)\}_{t=0}^\infty \) may be identified with the multipliers (or shadow prices) of the planner’s decision problem, whose intertemporal utility coincides with that of consumers.

**Proof of Theorem 6:**

Let \( \{c^*(t), S^*(t), K^*(t), \lambda^*(t), \phi^*(t)\}_{t=1}^\infty \) be a regular optimal path and denote by \( \{\chi^*(t)\}_{t=1}^\infty = \{c^*(t), S^*(t), \mu^*(t), K^*(t), \pi^*(t), r^*(t)\}_{t=1}^\infty \) the corresponding market-allocation path derived from it. (As above, we identify the interest rate \( r^*(t) \) with the supporting shadow price \( \nu^*(t) \) given by (75),

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the associated average profit \( \pi^*(t) \) then being obtained in the obvious fashion.

As outlined in the text, the basic idea underlying the proof of the Theorem is to have the planner use her commitment power in the game \( \mathcal{S} \) to select a strategy in the first stage of the game (i.e., a scheme \( \hat{\phi} \)) which guarantees that, in the second stage, only strategies that induce \( \{\chi^*(t)\}_{t=1}^\infty \) can be part of a continuation equilibrium. Since, by definition, such allocation path is maximal for the planner over all feasible paths, the selection of any such scheme \( \hat{\phi} \) must define an equilibrium of \( \mathcal{S} \). In particular, we may ignore the possibility that for other schemes \( \phi' \neq \hat{\phi} \) no continuation equilibrium may exit. Indeed, for this very same reason, any equilibrium of the game must be optimal, i.e., yield the same discounted payoff for the planner (and consumers).

We now construct some such scheme \( \hat{\phi} \). Given any arbitrary \( \chi = (c,S,\mu,K,\pi,r) \) define:

\[
\pi_n^*(\chi) = \max \{0, \pi_n^*(\mu,r)\}
\]

(77)

where \( \pi^*(\mu,r) \) was defined in Subsection 2.3.2. Consider the following scheme \( \hat{\phi}(\cdot) \), defined for all \( (\chi,t) \):

(i) For \( \chi = \chi^*(t) \), \( \forall n \geq 0 \), \( \hat{\phi}_n(\chi,t) = 0 \);

(ii) For \( \chi \neq \chi^*(t) \), \( \mu(\rho_0) < 1 \):

(a) If \( X(\chi) > 0 \), \( \forall n \geq 1 \), \( \hat{\phi}_n(\chi,t) = \pi_n^*(\chi) - \epsilon, 0 < \epsilon < X(\chi) \);

\[
\hat{\phi}_0(\chi,t) = \sum_{n \geq 1} \pi_n^*(\chi) \mu(\rho_n) + \epsilon(1 - \mu(\rho_0));
\]

(b) If \( X(\chi) = 0 \), \( \forall n \geq 0 \), \( \hat{\phi}_n(\chi,t) = 0 \);

(iii) For \( \chi \neq \chi^*(t) \), \( \mu(\rho_0) = 1 \): \( \forall n \geq 1 \), \( \hat{\phi}_n(\chi,t) = 0 \); \( \hat{\phi}_0(\chi,t) > 0 \).

It is easy to verify that the proposed scheme is feasible (i.e., satisfies (40) and (41)) and the only continuation equilibrium consistent with it must induce the optimal path \( \{\chi^*(t)\}_{t=1}^\infty \). Moreover, along such path, \( \xi(t) = 0 \), as claimed.
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