RATIONALITY OF BARGAINING SOLUTIONS

M. Carmen Sánchez*

WP-AD 96-09

*I would like to thank Josep E. Peris, B. Subiza, J. V. Llañeres, M. C. Marco and an anonymous referee of the series of working papers A Discussion (I. V. I. E) for their helpful comments and suggestions.

**University of Alicante.
RATIONALITY OF BARGAINING SOLUTIONS

M. Carmen Sánchez

ABSTRACT

We analyze the rationality of two person bargaining solutions by considering conditions which are weaker than those used by Peters and Wakker (1991) or Bossert (1994). As a particular consequence of their results, the rationality of the Nash solution is obtained, although they cannot be applied to other well known bargaining solutions. The aim of this paper is, on the one hand, to prove that a choice function defined on the usual bargaining domain which satisfies Independence of Irrelevant Alternatives, Weak Pareto Optimality and Pareto Continuity is also rationalized by a preorder (reflexive, complete and transitive binary relation). Moreover, the representability of this relation is analyzed. These results can be applied, in particular, to the Nash solution and moreover to the egalitarian (Kalai, 1977), monotone path solutions and their lexicographic extensions. On the other hand, and by substituting Pareto Continuity for Monotonicity assumptions, rationality is also analyzed. As a consequence, a result along the same lines as Bossert’s (1994) is obtained.

KEYWORDS: Rational Choice; Bargaining Solution.
0. INTRODUCTION

A bargaining solution is a special case of choice function which selects a unique outcome for each choice situation (bargaining problem) and which is achieved by means of cooperation by the agents involved. One of the interpretations of bargaining solutions is to consider that the outcomes of the bargaining problems are given by the recommendations of an "impartial arbitrator" whose preferences represent, in some sense, the preferences of the agents as a group. So, the agreement reached in each bargaining situation may be thought of as the most preferred alternative within the set of feasible outcomes according to the arbitrator's preferences.

Thus, rationality of bargaining solutions can be analyzed by making use of some results obtained for general rational choice functions. Along these lines we have to mention the works of Lenzberg (1987), Peters and Wakker (1991) and Bossert (1994) who provide a set of sufficient conditions in order to ensure the rationality of bargaining solutions by making use of standard axioms used in cooperative bargaining models (Pareto Optimality, Independence of Irrelevant Alternatives and Continuity). In order to do that, bargaining solutions are considered to be single-valued choice functions whose domain is given by the family of convex, comprehensive and compact subsets of $\mathbb{R}^n_+$ and, in this context, rationality by means of a transitive binary relation is analyzed.
Peters and Wakker (1991) assume that the domain of the choice function is given by the family of convex and compact subsets of $\mathbb{R}^2$, and prove that a single-valued choice function which satisfies Pareto Optimality, Pareto Continuity and Independence of Irrelevant Alternatives is rationalized by means of a transitive binary relation. Recently, Bossert (1994) has provided an alternative proof to this result by considering convex, compact and comprehensive subsets of $\mathbb{R}^2$ (which is the usual domain in bargaining theory) and by imposing Continuity instead of Pareto Continuity. As a particular consequence of this result, it is obtained that the Nash solution (1950) is rationalized by a preorder on the set of alternatives which are chosen in some bargaining problem.

As Bossert (1994) mentions in his work, there are other important bargaining solutions which are also rational (in the sense of choice functions) but whose rationality is not obtained by applying that result. This is the case, for instance, of the egalitarian solution (Kalai, 1977), monotone path solutions and the lexicographic extensions of these solutions which satisfy axioms which are different to those used in Peters and Wakker's or Bossert's works (axiomatic characterizations of these solutions can be found in Thomson, forthcoming).

In this paper it is proved that a single-valued choice function defined in the usual bargaining domain which satisfies Pareto Continuity, Weak Pareto Optimality and Independence of Irrelevant Alternatives is rational, generalizing Bossert's result (1994). The existence of a "numerical representation" of the choice function is also analyzed.

Moreover, it is also proved that, in the same context, a choice function which satisfies Independence of Irrelevant Alternatives and Strong Monotonicity is rational on the set of alternatives chosen in some choice situation. Finally we present a different rationality result by weakening the monotonicity assumption. In particular it is proved that a choice function which satisfies Interior Monotonicity, Independence of Irrelevant Alternatives and Pareto Optimality is also rational. As a consequence of all of these results the rationality of the egalitarian, monotone path solutions as well as their lexicographic extensions is obtained.

The paper is organized as follows: in Section 1 notation and assumptions which will be used through the paper are introduced. In Section 2 rationality results are presented and applied in order to obtain the rationality and representability of well known bargaining solutions and finally, in Section 3, rationality results without continuity are obtained.
1. PRELIMINARIES.

A bargaining problem is usually described by a pair \((S, \delta)\) where \(S \subseteq \mathbb{R}_+^n\) is the family of feasible utility vectors which individuals can afford and \(d \in \mathbb{R}_+^n\) represents the disagreement point, that is the vector of utilities which individuals would obtain if they do not reach an agreement. As is usual in bargaining theory we assume that \(d = (0, 0)\) and we restrict our attention only to subsets of \(\mathbb{R}_+^n\) (gains of the agents over the disagreement point).

Moreover we consider the usual domain in bargaining theory which is given by

\[
D = \{ A \subseteq \mathbb{R}_+^n \mid A \text{ is convex, compact, comprehensive and there exists } x \in A \text{ such that } x > 0 \}
\]

where by \(x > 0\) we mean that \(x_i > 0\) \(\forall i = 1, 2\). From now on, by \(x \geq y\) we mean \(x_1 \geq y_1 \forall i = 1, 2\) and by \(x \geq y, x_i \geq y_i \forall i = 1, 2\).

In order to simplify notation, for every subset \(A\) of \(\mathbb{R}_+^n\) we will denote by \(\langle A \rangle\) the comprehensive and convex hull of \(A\), which is defined in the usual way, that is

\[
\langle A \rangle = \cap \{ B \subseteq \mathbb{R}_+^n \mid A \subseteq B, B \text{ convex and comprehensive} \}
\]

Formally a bargaining solution is a function which selects a unique outcome for each and every subset of outcomes which is considered (bargaining problem) within the domain \(D\) defined above. That is

\[
F: D \rightarrow \mathbb{R}_+^n \\
\forall S \in D \quad F(S) \in S
\]

We will denote by \(F(D)\) the set of alternatives which are chosen in some bargaining problem by the solution \(F\), that is

\[
F(D) = \{ a \in \mathbb{R}_+^n \mid \exists S \in D \text{ such that } F(S) = a \}
\]

Therefore a bargaining solution is a particular case of single-valued choice function in which the universal set of alternatives is considered to be \(\mathbb{R}_+^n\) and whose domain is the usual one in bargaining theory as defined previously. From now on, this is the context in which we are going to work.

The notion of rationality of a (single-valued) choice function states the existence of some kind of binary relation whose maximization determines the choice set for each choice situation. Formally, \(F\) is rational if there exists a reflexive, complete and transitive binary relation \(R\) such that \(\forall S \in D\),

\[
|F(S)| = \{ a \in \mathbb{R}_+^n \mid a R z \quad \forall z \in S \}
\]
This notion of rational choice function corresponds to the notion of regular-rational choice function used by Richter (1971), who proves that this kind of rationality is equivalent to the transitive one in the single-valued case.

Next we present the different assumptions (all of them standard ones in bargaining theory) which will be used to obtain the rationality results.

\[(PO). \text{Pareto Optimality:}\]
\[
\forall S \in D, \quad F(S) \in \text{PO}(S) = \{ x \in S \mid y \geq x, y \neq x \Rightarrow y \in S \}
\]

\[(WPO). \text{Weak Pareto Optimality:}\]
\[
\forall S \in D, \quad F(S) \in \text{WPO}(S) = \{ x \in S \mid y > x \Rightarrow y \in S \}
\]

\[(IIA). \text{Independence of Irrelevant Alternatives:}\]
\[
\forall S_1, S_2 \in D, \quad S_1 \subseteq S_2 \quad \text{and} \quad F(S_2) \in S_1 \Rightarrow F(S_1) = F(S_2)
\]

It is important to point out that (IIA) is necessary for the rationality of a single-valued choice function (Peters and Wakker, 1991), so any bargaining solution which does not satisfy this assumption can not be rational in the sense we have just defined above. For instance, this is the case of the Kalai-Smorodinsky solution (Kalai and Smorodinsky, 1975), the Equal-Loss solution (Chun, 1988) or the Rational Equal-Loss solution (Herrero and Marco, 1993).

With respect to continuity assumptions, we will make use of \textit{Pareto Continuity} (Peters, 1986) which is the condition also used by Peters and Wakker (1991) to present their results. This condition is weaker than usual \textit{Continuity} (slight changes in the subset of alternatives presented for choice imply that the solution outcome does not change radically) and it takes into account not only changes in the set of alternatives presented for choice but also in its Pareto Optimal boundary.

\[(CONT). \text{Continuity: If} \{A_n\} \subset D, \{A_n\} \rightarrow \{A\} \text{in the Hausdorff topology, then} \{F(A_n)\} \rightarrow \{F(A)\}.\]

\[(PC). \text{Pareto Continuity: If} \{A_n\} \subset D, \{A_n\} \rightarrow \{A\} \text{and} \{\text{PO}(A_n)\} \rightarrow \{\text{PO}(A)\} \text{in the Hausdorff topology, then} \{F(A_n)\} \rightarrow \{F(A)\}.\]

The binary relation which will be used to obtain the rationality of bargaining solutions is the \textit{revealed preference relation} which is formally defined as follows,

\[
\forall x, y \in \mathbb{R}_+, \quad x \not\sim y \quad x \not\succeq y \Rightarrow \exists A \in D \mid x, y \in A \quad \text{and} \quad F(A) = x
\]

So this is an irreflexive binary relation although it is not generally complete. From this we define the \textit{transitive closure of the revealed preference (P*)} as follows,

\[
x \not\sim y \Rightarrow \exists x_1, x_2, \ldots, x_n \in X \text{ such that } x = x_1 \not\sim x_2 \not\sim \ldots \not\sim x_n = y
\]

The non-existence of cycles of length 2 for the revealed preference is known as the \textit{Weak Axiom of the Revealed Preference (WARP)}, which is
equivalent in our context (domain closed under intersection) to (IIA) (Hansson, 1968). Moreover whenever (IIA) is imposed we can use the following equivalent formulation of the revealed preference relation which will be used in the proofs:

$$\forall x, y \in \mathbb{R}^+_n \quad x \succ y \quad x P y \quad \iff \quad F(x, y) = x$$

In order to obtain rationality results for bargaining solutions we will make use of the well-known Strong Axiom of Revealed Preference (SARP) (acyclicity of the revealed preference). Formally,

(SARP). Strong Axiom of Revealed Preference: The revealed preference relation does not have cycles, that is

$$x_1 P x_2 P \ldots P x_n \implies \text{no}[x_n P x_1]$$

Ville (1946), and independently Houthakker (1950), proved that, in general, this axiom is equivalent to rationality by means of a transitive binary relation. However, in the particular case of considering single-valued choice functions, the Strong Axiom of Revealed Preference also characterizes the rationality by means of a reflexive, transitive and complete binary relation [see Richter (1971); Corollary 1, Chapter 2]. Therefore, in order to obtain some of the rationality results, it will in fact be proved that the Strong Axiom of Revealed Preference is satisfied.

2. RATIONALITY RESULTS.

Peters and Wakker's (1991) or Bossert's (1994) results can be applied in order to obtain (in different contexts) that the Nash solution is rational. If we wish to analyze the rationality of other well-known solutions, then either Pareto Optimality or Continuity conditions must be removed. The first result of the paper makes use of Weak Pareto Optimality instead of Pareto Optimality and proves that this assumption together with Independence of Irrelevant Alternatives and Pareto Continuity also implies rationality. In particular this result generalizes the one obtained by Bossert (1994) and allows us to prove the rationality of any monotone path solutions as well as of their lexicographic extensions.

THEOREM 1. If $F: \mathcal{D} \rightarrow \mathbb{R}^+_n$ is a choice function which satisfies (IIA), (WPO) and (PC), then it satisfies (SARP).

Proof. (See Appendix A).

Remark 1: It is important to note that, although the usual bargaining solutions satisfy at least Weak Pareto Optimality, the previous result can also be proved by dropping this condition and by imposing only Independence of Irrelevant Alternatives and Pareto Continuity. Of course, some of the arguments used in the way of reasoning should change. So, although (IIA) is not enough to ensure the rationality of bargaining solutions (see Peters and Wakker, 1991), if we add (PC), this rationality can be obtained.
As we mentioned above and as a consequence of this result, not only the Nash solution but any bargaining solution satisfying (PC), (WPO) and (IIA) are rational, which we formulate as a corollary.

**COROLLARY 1.** Nash, Egalitarian, monotone path solutions and their lexicographic extensions are rational.

Now we want to know some properties of the relation which rationalizes these bargaining solutions. Along these lines, Peters and Wakker analyze the "representability" of the binary relation: a function \( f : \mathbb{R}_+ \to \mathbb{R} \) represents a binary relation \( P \), if \( \forall x, y \in \mathbb{R}_+ \quad x \ P \ y \) implies \( f(x) > f(y) \). In particular they prove the existence of a function which represents the transitive closure of the revealed preference associated to a choice function \( F \) (which satisfies (CONT), (PO) and (IIA)) and which is maximized by \( F \), that is \( f(F(A)) > f(x) \forall A \in D, x \in A, x \neq F(A) \).

The following result shows that this representability result can be also obtained in the context we have considered if Pareto Optimality conditions are removed. Therefore, not only the Nash solution works as if maximizing a group utility function, but also the egalitarian and monotone path solutions.

**THEOREM 2.** If \( F \) is a choice function which satisfies (CONT) and (IIA), then there exists a real function \( f : \mathbb{R}_+ \to \mathbb{R} \) such that \( F \) maximizes \( f \) over \( D \).

**Proof.** Consider the following subset,

\[
A = \{ a \in \mathbb{R}_+ \mid a = P(<z>) \text{ for some } z \in \mathbb{R}_+^2 \}
\]

Since it is a countable set, we rename its elements as follows,

\[
A = \{ a_i \}_{i \in I}, \quad I \neq \emptyset, \ I \subseteq \mathbb{N}
\]

Now by considering the transitive closure of the revealed preference (which is asymmetric by applying Remark 1) we define, for each alternative \( x \in \mathbb{R}_+ \), the set \( I_x = \{ i \in I \mid x \ P^* a_i \} \) and the following function:

\[
f(x) = \begin{cases} 
\sum_{i \in I_x} \frac{1}{2^i} & \text{if } I_x \neq \emptyset \\
0 & \text{if } I_x = \emptyset
\end{cases}
\]

First we are going to prove that \( x \ P \ y \) implies \( f(x) > f(y) \). Consider \( x, y \in \mathbb{R}_+ \) such that \( x \ P \ y \). By (IIA) we know that \( F(<x,y>) = x \) and \( F(<y,x>) = x \). Moreover it is clear that if \( x \ P^* y \) then \( I_y \subseteq I_x \) and therefore \( f(x) \geq f(y) \). We analyze two different cases.

**Case 1.** \( y \in F(D) \).

In this case and by definition of \( P \), we know that there does not exist \( z \in \mathbb{R}_+ \) such that \( y \ P \ z \). In particular, there does not exist \( z \in A \) such that \( y \ P^* z \), therefore \( I_y = \emptyset \) and \( f(y) = 0 \).
With respect to alternative $x$, since $F(<x>) = x$, we know that $x \in P$ for all $a \in <x>$. So if we consider $b \in <x> \cap \Theta_a^i$ and we denote $F(cb) = w$, then we obtain that $w \in A$ and $x \in P w$, so we can ensure that $f(x) > 0$.

Case 2. $y \in F(D)$.

From (IA) this implies that $F(y) = y$. We are going to analyze the different configurations of alternatives $(x, y)$.

If $x \leq y$, then $x \in <y>$ and we obtain $y \in P x$, contradicting the asymmetry of $P$.

If $y \leq x$ and $y \in \text{int}(<x>)$, then it is enough to consider an alternative $z \in \text{int}(<x>) \cap \Theta_a^i$ such that $y \in \text{int}(<z>)$. Since $F(<z>) \in A$, we denote $F(<z>) = a_j$, thus we obtain that $j \in I_a$ due to $a_j \in <z> \subseteq <x>$, but $j \notin I_a$, since $y \in <z>$, therefore if $a_j = y$ it implies that $a_j \in P y$ and if not ($a_j = y$) then no($y \in P a_j$) from the definition of $P$. So we can conclude that $f(x) > f(y)$.

In other cases, that is if $(y \leq x$ with $y \in <x> \cap \text{int}(<x>)$) or [no($x \leq y$) and no($y \leq x$)], we can consider a sequence of alternatives as follows,

\[ \{x_1\} \in \Theta_a^i \cap [\text{int}(<x,y>)<y>] \quad [x_i] \rightarrow x \]

By applying the continuity of $F$ we can ensure on the one hand that

\[ F(y,z_j) \longrightarrow F(x,y) = x \]

and, on the other hand, that there exists $k \in \mathbb{N}$ such that

\[ F(y,z_j) \in \{x,y\} \]

since $x \in <y,z_j> \forall j \in \mathbb{N}$ and if $F(<y,z_j>) = y \forall j$ we obtain a contradiction with the continuity of $F$. Therefore if we denote $b_k = F(<y,z_k>)$ we have that $b_k \in P y$. If we now consider an alternative $s \in \Theta_a^i \cap <x>$ such that $s > b_k$ and denote $F(s) = a_k$, we obtain that $a_k \in A$ and $x \in P a_k \in P b_k \in P y$, so $x \in P a_k \in P y$ and therefore $f(x) > f(y)$.

Finally it is clear that $F$ maximizes $f$: if $S \in D$ and $F(S) = w$, then $w \in P S \forall s \in S \rightarrow w$ and therefore $f(w) > f(s) \forall s \in S \rightarrow w$.  \qed
3. RATIONALITY RESULTS WITHOUT CONTINUITY.

In this Section we present two different rationality results by dropping continuity conditions and by imposing monotonicity assumptions instead. The first one is along the same lines as Bossert’s result (1994) in which it is proved that any choice function which satisfies (CONT), (IIA) and (PO) is not only rational on \( F(D) \), but the rationalization satisfies some regularity conditions (monotonicity, strict quasi-concavity,...). Along these lines we will prove that a choice function which satisfies Independence of Irrelevant Alternatives and Strong Monotonicity (if we expand a set, then the outcome selected by the bargaining solution in the new set is greater than or equal to the initial one) is also rational on \( F(D) \) and that, under these assumptions, the rationalization satisfies monotonicity and strict quaiconcavity. So we drop Pareto Optimality and Continuity, although we need to require Strong Monotonicity in their place. Formally the monotonicity assumption is as follows:

\[
(SM). \text{Strong Monotonicity:} \\
\forall A, B \in D \text{ such that } A \subseteq B, \quad F(A) \leq F(B)
\]

**THEOREM 3.** If \( F \) is a choice function satisfying (IIA) and (SM), then it is rational on \( F(D) \).

**Proof.** First we are going to prove that the revealed preference \( \mathcal{P} \) is transitive and then it will in fact be proved that it is complete when we restrict our attention to \( F(D) \). As in the previous cases, by (IIA) we obtain the asymmetry of the revealed preference.

Assume that \( x \mathcal{P} y \mathcal{P} z \) for some \( x, y, z \in \mathbb{R}^2 \). Then by (IIA) we know that \( F(x, y) = x, F(y, z) = y, F(x, y) = x \) and \( F(x, y) = y \). We consider now the subset \( x, y, z \) and analyze \( F(x, y, z) = w \). If \( w = x \), then \( x \mathcal{P} z \) and we obtain the transitivity of \( \mathcal{P} \).

By contradiction, assume that \( w \neq x \). If \( w = y \) then \( y \mathcal{P} x \), contradicting the asymmetry of \( \mathcal{P} \). If \( w = z \) then \( z \mathcal{P} y \) contradicting the asymmetry of \( \mathcal{P} \) once again. If \( w \in \langle x, y \rangle \) then by (IIA) \( w = x \), a contradiction. If \( w \in \langle y, z \rangle \) then by (IIA) \( w = y \), a contradiction. Therefore the only possibility is that \( w \in \langle x, z \rangle \), so by (IIA) \( F(x, z) = w \). We now analyze the configuration of \( \langle x, z \rangle \). If \( x \leq z \) then \( x, y, z \rangle = \langle y, z \rangle \), a contradiction since \( w \neq y \); if \( z \leq x \) then \( \langle x, z \rangle = \langle x \rangle \), a contradiction since \( w \neq x \). Therefore \( x \leq z \) and \( z \leq x \), but by applying Strong Monotonicity we know that \( y \leq x \langle x, y \rangle \leq \langle x, y \rangle \), therefore the only possibility is that \( x = w \), a contradiction.

To show that \( \mathcal{P} \) is complete on \( F(D) \), consider \( x, y \in F(D) \), \( x \neq y \). It is clear that it implies that \( x \leq y \) or \( y \leq x \): if \( x \leq y \) and \( : y \leq x \) and we denote by \( w = F(x, y) \), we know (by applying Strong Monotonicity) that since \( \langle x \rangle \leq \langle x, y \rangle \), \( \langle y \rangle \leq \langle x, y \rangle \) and by (IIA) \( F(x, y) = x \) and \( F(x, y) = y \), then \( x \leq w \) and \( y \leq w \), which implies that \( w \in \langle x, y \rangle \), a contradiction. But if \( x \leq y \) then \( x \in \langle x \rangle \), so \( y \mathcal{P} x \) and if \( y \leq x \) then \( y \in \langle x \rangle \) and therefore \( x \mathcal{P} y \).
Finally and in order to ensure the reflexivity we modify this relation by considering
\[ x \, R \, y \iff x = y \text{ or } x \, P \, y \]
which is obviously complete, reflexive, transitive and also a rationalization of \( F \) on \( F(D) \).

With respect to the additional properties which this rationalization on \( F(D) \) satisfies, we can ensure that under the same assumptions [(IIA) and (SM)] it satisfies Monotonicity and Strict Quasi-concavity. First we formally present these properties.

**Monotonicity:** \( \forall x, y \in F(D), \text{ if } x > y \text{ then } x \, P \, y \).

**Strict Quasi-concavity:** \( \forall x, y \in F(D) \text{ if } x \, P \, y \text{ and for some } \alpha \in (0,1) \)
\[ [\alpha x + (1-\alpha)y] \in F(D), \text{ then } \alpha x + (1-\alpha)y \, P \, y. \]

**THEOREM 4.** If \( F \) is a choice function satisfying (IIA) and (SM), then there exists a rationalization on \( F(D) \) which satisfies Monotonicity and Strict Quasi-concavity.

**Proof.** By applying Theorem 3 we know that there exists a rationalization on \( F(D) \) given by \( R (x \, R \, y \iff x = y \text{ or } x \, P \, y) \). We only need to show that this relation satisfies the conditions mentioned above.

**Monotonicity:** Consider \( a, b \in F(D) \) such that \( a > b \). Since \( a, b \in F(D) \) we know by (IIA) that \( F(<a>) = a \) and \( F(<b>) = b \); so \( b \in <a> \) and therefore \( a \, P \, b \).

**Strict Quasi-concavity:** Consider \( a, b \in F(D) \) such that \( a \, P \, b \) and \( \alpha \in (0,1) \) such that \( [\alpha a + (1-\alpha)b] \in F(D) \). By (IIA) we know that \( F(<a>) = a \) and \( F(<b>) = b \), and \( F(<\alpha a + (1-\alpha)b>) = \alpha a + (1-\alpha)b \). Moreover and by reasoning as in Theorem 3 we know that \( a \leq b \) or \( b \leq a \); but \( a \leq b \) implies that \( a \in <b> \) and therefore \( b \, P \, a \), in contradiction with the asymmetry of \( P \). So the only possibility is \( b \leq a \), which implies \( \alpha a + (1-\alpha)b \geq b \), \( b \in <\alpha a + (1-\alpha)b> \) and therefore \( \alpha a + (1-\alpha)b \, P \, b \).

**Remark 2:** It is easy to prove that if we impose continuity to the choice function then the rationalization is also **Upper Semicontinuous** (that is \( \forall x \in F(D), [y \in F(D) \mid x \, P \, y] \text{ is open in } F(D)\)).

As an immediate consequence of this result it is also obtained that egalitarian and monotone path solutions are rationalized by an order on \( F(D) \) satisfying those conditions.

**COROLLARY 2.** Egalitarian and Path monotone solutions are rational on \( F(D) \) and the rationalization satisfies Monotonicity, Strict Quasi-concavity and Upper Semicontinuity.

In order to obtain an alternative characterization result without considering continuity conditions and by weakening the monotonicity
assumption, we present the following result. First we need the notion of Interior Pareto Optimal point and Interior Monotonicity.

If $S \in D$, an alternative $x \in S \subseteq \mathbb{R}^2$ is an interior Pareto optimal point if it is satisfied that $x \in \text{int}[\text{PO}(S)]$, where by $\text{int}[\text{PO}(S)]$ we mean the relative interior of subset $\text{PO}(S)$.

So the notion of Interior Monotonicity in the two-dimensional case states that if given a subset of alternatives, the outcome solution is an interior Pareto optimal point, then if we expand the feasible set, the new solution has to be greater than or equal to the initial one. Formally,

(IM). Interior Monotonicity: $\forall A, B \in D$, if

$$A \subseteq B \quad \text{and} \quad F(A) \text{ is an interior Pareto optimal point in } A \} \quad \Rightarrow \quad F(A) \leq F(B)$$

The next result shows that if we require Interior Monotonicity together with Independence of Irrelevant Alternatives and Pareto Optimality of the choice function, then it is obtained that the revealed preference is acyclic.

THEOREM 5. If $F: D \rightarrow \mathbb{R}^2$ is a choice function which satisfies (PO), (IIA) and (IM), then it satisfies (SARP).
APPENDIX A

In order to simplify the notation, from now on whenever we write \( i + j \) we mean \( i + j \) if \( i + j \leq n \) and \( i + j - n \) if \( i + j > n \) for every \( i, j \in \{1, 2, ..., n\} \). Moreover, \( \forall A \in D, F(A) \) denotes the topological boundary of \( A \) and for any pair of alternatives \( x, y \in \mathbb{R}^2 \), \( \text{seg}(x, y) \) denotes the segment which joins \( x \) and \( y \) up without considering \( x \) and \( y \) (in another case we use \( \text{seg}(x, y) \)), that is, \( \text{seg}(x, y) = \{\lambda x + (1-\lambda)y \mid \lambda \in (0,1)\} \).

To prove the acyclicity of the revealed preference we are going to apply induction, therefore first we present the following result obtained by Bossert (1994) in which the non existence of cycles of length 3 is proved. In particular it shows that (IIA) is enough to guarantee the non existence of cycles of length 3.

**Theorem 6.** [Bossert, 1994] If \( F:D \rightarrow \mathbb{R}^2 \) is a univalued choice function which satisfies (IIA), then there do not exist cycles of length 3 for the revealed preference.

An analogous result was obtained by Peters and Wakker (1991) in the context of convex and compact (but not necessarily comprehensive) subsets of \( \mathbb{R}^2 \), but by imposing (PO) apart from (IIA) (see Peters and Wakker (1991); Lemma 3.5). Moreover they prove that (IIA) and (PO) are not enough to guarantee the non-existence of cycles of length greater than 3.

**Proof of Theorem 1.** We prove the acyclicity of the revealed preference relation by induction. Because (IIA) is equivalent to (WARP) in this context, there do not exist cycles of length 2 and, by Theorem 6, there do not exist cycles of length 3. Let \( n \geq 4 \) and suppose that there exists no cycle of length \( k \) for all \( k < n \); we are going to prove that there are no cycles of length \( n \) either. By way of contradiction suppose that there exists a cycle of length \( n \), that is \( x_1, x_2, ..., x_n \in \mathbb{R}^2 \) such that \( x_i \leq F(x_i, x_{i+1}) \) by (IIA):

\[
F(x_i, x_{i+1}) = x_i \quad \forall i = 1, 2, ..., n
\]

Moreover it is clear that \( x_i \) cannot be in \( x_j \) for all \( j \neq i \) (if not, there exists a cycle of a length smaller than \( n \)) and therefore

\[
\{ x_i^1 > x_j^1 \text{ and } x_i^2 < x_j^2 \} \text{ or } \{ x_i^1 < x_j^1 \text{ and } x_i^2 > x_j^2 \}
\]

for all \( i, j \in \{1, 2, ..., n\} \). Furthermore and in general, we can also ensure that

\[
x_j \not\in F(x_i, x_{i+1}) \quad \forall i = 1, 2, ..., n, \ j \neq p, p+1
\]

since if not we obtain a cycle of length smaller than \( n \). For the same reason if we denote \( A = \{x_1, x_2, ..., x_n\} \) we can ensure that whenever we consider the comprehensive and convex hull of a subset of alternatives of \( A \) with more than two alternatives or with two non consecutive ones according
to the cycle, then the choice set will never be one of the alternatives of
the subset, that is

\[ \forall B \subseteq A \text{ such that } B \text{ has more than 2 alternatives or two }
\]

non consecutive ones, then \( F(B) \notin B \) \hspace{1cm} (2)

If we now analyze the choice over the convex and comprehensive hull
of \( A, w = F(A) \), it must be \( w \neq x_i \forall i=1,\ldots,n \), since if not, we obtain a
cycle of length 2, a contradiction. So there exist alternatives \( x_{i1}, x_{i+n} \in A, \)

\( x_i \neq x_{i+n} \) such that \( w = \langle x_{i1}, x_{i+n} \rangle \). Moreover and by (IIA) these alternatives
are non consecutive, since if \( w \in \langle x_{i1}, x_{i+n} \rangle \) for some \( i \) then it is obtained
that \( F(x_{i1}, x_{i+n}) = w \), a contradiction. Finally note that \( w \in \text{seg}(x_{i1}, x_{i+n}) \)
since if not by (WPO) either \( w \notin \langle x_{i1} \rangle \) or \( w \notin \langle x_{i+n} \rangle \) and by (IIA) it would
imply that \( F(x_{i1}) = w \) or \( F(x_{i+n}) = w \) respectively, a contradiction.

Since we start from a cycle, we can assume without loss of generality
that \( x_{i1} < x_{i+n} \), so the configuration of these two alternatives would be as
Fig.1 shows.

Next we are going to analyze the different cases which exhaust all
possible configurations of alternatives \( \{x_{i1}, x_{i+n}, \ldots, x_{i+n+k}\} \). Since by
(WPO) \( \text{seg}(x_{i1}, x_{i+n}) \in F(A) \) we know that any other alternative of \( A \)
(\( x_j \in A, j \neq i \)) cannot be placed over the line which passes through \( x_j \)
and \( x_{i+n} \) (see Fig.1). So, and from the considerations we have made above,
we can ensure that there are at least three possible positions to place
alternative \( x_{i1} \), which have been denoted as \( A_i, B_i \) and \( C_i \) (see Fig.1).

CASE I: \( x_{i1} \in A_i \).

In this case by applying (2), (IIA) and (WPO) we know that
\( F(x_{i1}, x_{i+n}) = w \in \text{seg}(x_{i1}, x_{i+n}) \), while applying (IIA) it is satisfied
that \( F(x_{i1}, x_{i+n}, x_{i+n+k}) = w \) (since \( x_{i1}, x_{i+n}, x_{i+n+k} \subset A \)) and
\( F(A) \in \langle x_{i1}, x_{i+n}, x_{i+n+k} \rangle \). We are going to prove that this fact contradicts
continuity.

Consider the configuration of alternatives \( w, x_{i+n}, x_{i+n+k} \) (see Fig.2).

For any \( t \in [0,1] \) we define \( z(t) = tw + (1-t)x_{i+n} \),
\( A(t) = \langle x_{i1}, x_{i+n}, z(t) \rangle \) and the following function:
\( g: [0, 1] \rightarrow \mathbb{R}^2 \) such that \( g(t) = F(A(t)) \)

This is a continuous function since it is a composition of continuous ones, and moreover by (IIA) and (WPO) it is clear that \( \forall t \in [0, 1] \), \( g(t) = F(A(t)) \notin A(0) \) or \( g(t) = w_i \). We denote \( B = g^{-1}(w_i) \), which is closed (due to the continuity of \( g \)), nonempty \( (0 \in B) \) and \( B \neq [0, 1] \) \( (1 \in B) \). Therefore we can ensure the existence of maximum of \( B \), namely \( t_0 \). Since \( t_0 \in B \) we know that \( g(t_0) = w_i \) and by definition of maximum \( \forall t \in [0, 1] \) such that \( t > t_0 \) it is satisfied that \( g(t) \neq w_i \). But this is a contradiction with the continuity of \( g \) by considering \( \varepsilon < d(w_i, x_{i,k}) \), since \( g(t) \notin A(0) \) or \( g(t) = w_i \) and therefore we can consider \( t \) as close as we want to \( t_0 \) but \( d(x(t), x(t_0)) < \varepsilon \) \( \forall t > t_0 \).

**CASE 2:** \( x_{i,1} \in B_1 \).

Given the configuration of alternatives \( x_{i,1}, x_i \) and \( x_{i,k} \) (see Fig.3), we analyze the possible location of alternative \( x_{i,k} \). The different possibilities for locating this alternative have been denoted as \( B_2, C_2, D_2, E_2 \) and \( F_2 \). We are going to prove that the only one which does not give us a contradiction is \( D_2 \).

2.1. If \( x_{i,2} \in A_2, B_2, C_2, \) or \( F_2 \), then by reasoning in an analogous way to the previous case, by considering the following subsets in each case, a contradiction with continuity is obtained (see Fig.4 and Fig.5):

In case of \( x_{i,2} \in A_2 \), consider \( x_{i,2}, x_{i,2} \) and \( x_{i,2} \); if \( x_{i,2} \in B_2 \), consider \( x_{i,2}, x_{i,2} \), \( F(x_{i,2}, x_{i,2}, x_{i,2}) \) and \( x_{i,2}, x_{i,2} \); if \( x_{i,2} \in C_2 \), consider \( x_{i,2}, x_{i,2} \), \( w_{i,2} \) and \( x_{i,2}, x_{i,2} \); and, finally, if \( x_{i,2} \in D_2 \), consider \( F(x_{i,2}, x_{i,2}, x_{i,2}, x_{i,2}) \) and \( x_{i,2}, x_{i,2} \).

---

28

---

29
2.2. Finally if \( x_{ir2} \in E_2 \), then \( F(<x_{ir2},x_{ih1},x_{ir3}>) \) belongs to \( <x_{ih1}^*,x_{ir3}^*> \cup <x_{ir3}^*,x_{ir2}^*> \), which implies by (IIA) a contradiction with (2).

Therefore, as we mentioned above, the only possibility for alternative \( x_{ir3} \) is \( D_3 \). The final configuration of alternatives \( x_i, x_{ir1}, x_{ir2} \) and \( x_{ir3} \) is shown in Fig.6.

![Fig. 6](image)

By following an analogous way of reasoning, we are going to show that there is also a unique possibility for locating alternative \( x_{ir3} \) which has been denoted as \( F_3 \) (see Fig.6). Notice that this zone is defined by the line which passes through alternatives \( x_{ir2} \) and \( x_{ir3} \) and the line which goes through alternatives \( x_{ir2} \) and \( x_{ir3} \). Therefore the last two previous alternatives which have been located together with \( x_{ir3} \) are the references to determine which this area is.

2.3. If \( x_{ir3} \) belongs to \( A_3, C_3, E_3 \) and \( G_3 \), a contradiction with continuity is obtained [by considering the following subsets in each case: if \( x_{ir3} \in A_3 \), consider \( <x_{ir3}^*,x_{ir3}^*> \) and \( <x_{ir3}^*,x_{ir3}^*> \); if \( x_{ir3} \in C_3 \), consider

\( <x_{ir3}^*,x_{ir3}^*> \) and \( <x_{ir3}^*,x_{ir3}^*> \); if \( x_{ir3} \in E_3 \), consider \( <x_{ir3}^*,x_{ir3}^*> \) and \( <x_{ir3}^*,x_{ir3}^*> \); if \( x_{ir3} \in G_3 \) then consider \( <x_{ir3}^*,x_{ir3}^*> \) and \( <x_{ir3}^*,x_{ir3}^*> \)].

2.4. If \( x_{ir3} \in B_3 \) then \( x_{ir3} \in <x_{ir3}^*,x_{ir3}^*> \), a contradiction.

2.5. Finally if \( x_{ir3} \in D_3 \) then \( F(<x_{ir3}^*,x_{ir3}^*>) \) belongs to \( <x_{ir3}^*,x_{ir3}^*> \cup <x_{ir3}^*,x_{ir3}^*> \), in contradiction with (2) by (IIA).

Therefore, and by following an analogous argument for the rest of alternatives \( (x_{ir3}^*,x_{ir3+1},...x_{ir3+k}) \) we can ensure that the final configuration of those alternatives in this case would be as shown in Fig.7.

![Fig. 7](image)

The final contradiction which is obtained is that there does not exist any possibility of locating alternative \( x_{ir3+k+1} \). On the one hand we know that the only possibilities for locating this alternative are the areas denoted as \( A_4, B_4, C_4, D_4 \) (see Fig.7). But in all of these cases a contradiction is obtained. In particular if \( x_{ir3+k+1} \in A_4 \) then \( F(<x_{ir3+k+1},x_{ir3+k+1}>) = w \) and \( F(<x_{ir3+k+1},x_{ir3+k+1}>) \in \text{seg}(x_{ir3+k+1},x_{ir3+k+1}) \), in contradiction with continuity; if \( x_{ir3+k+1} \in D_4 \) then \( F(<x_{ir3+k+1},x_{ir3+k+1}>) \)
belongs to $<x_{i+1}, x_{i+2} > \cup <x_{i+2}, x_{i+3} >$ in contradiction with (2) by (IA); if $x_{i+1} \in B_4$ then some of the alternatives $x_{ij}$ for $j=1,2,\ldots,k-1$ belongs to $<x_{i+1}, x_{i+2} >$ a contradiction; and finally, if $x_{i+1} \in C_4$ then by considering $<x_{i+1}, x_{i+2} >$ a contradiction with continuity is obtained once again $(F(x_{i+1}, x_{i+2}) \in \text{seg}(x_{i+1}, x_{i+2}))$ and $F(x_{i+1}, x_{i+2}) = x_{i+1}$).

Therefore in this case a contradiction is also obtained and so the only possibility for the final configuration of alternative $x_{i+1}$ is $C_1$ (see Fig.1).

CASE 3: $x_{i+1} \in C_1$.

The different possibilities for alternative $x_{i+2}$ have been denoted as $A_5$, $B_5$, $C_5$, $D_5$ and $E_5$ (see Fig.8).

But in $A_5$, $B_5$, $C_5$ and $E_5$ a contradiction with continuity is obtained by reasoning in a similar way to the previous cases [in particular by considering the following subsets in each case: if $x_{i+2} \in A_5$, consider $<x_{i+2}, x_{i+3} >$ and $<x_{i+3}, x_{i+4} >$; if $x_{i+2} \in B_5$, consider $<x_{i+2}, x_{i+3} >$ and $<x_{i+3}, x_{i+4} >$; and $<x_{i+4}, x_{i+5} >$; if $x_{i+2} \in C_5$, consider $<x_{i+2}, x_{i+3} >$ and $<x_{i+3}, x_{i+4} >$; and finally, if $x_{i+2} \in E_5$ then consider $<x_{i+2}, x_{i+3} >$ and $<x_{i+3}, x_{i+4} >$.]

So the only possibility for alternative $x_{i+2}$ is $D_5$ (see Fig.9), but by reasoning as in case 2 the only possibility for $x_{i+3}$ is $C_6$ and the final configuration of alternatives $x_1, x_2, \ldots, x_n$ is shown in Fig.10. A contradiction is now obtained when we try to locate alternative $x_{i+1}$ (it can not be located in any place).

Therefore we can conclude that there do not exist cycles of length $n$ and the Strong Axiom of the Revealed Preference is satisfied.  

Therefore we can conclude that there do not exist cycles of length $n$ and the Strong Axiom of the Revealed Preference is satisfied.  

Therefore we can conclude that there do not exist cycles of length $n$ and the Strong Axiom of the Revealed Preference is satisfied.  

Therefore we can conclude that there do not exist cycles of length $n$ and the Strong Axiom of the Revealed Preference is satisfied.  

Therefore we can conclude that there do not exist cycles of length $n$ and the Strong Axiom of the Revealed Preference is satisfied.  

Therefore we can conclude that there do not exist cycles of length $n$ and the Strong Axiom of the Revealed Preference is satisfied.
APPENDIX B

In order to simplify the proof of Theorem 5, first we present the following lemma in which the non existence of cycles of length 4 is proved by requiring (PO), (IIA) and (IM).

**LEMMA 1.** If \( F : D \rightarrow X^2 \) is a univalued choice function which satisfies (PO), (IIA) and (IM), then there do not exist cycles of length 4 for the revealed preference relation.

**Proof:** By contradiction, assume that there exist alternatives \( x_1, x_2, x_3, x_4 \in X \) such that \( x_1 \succ_i x_2 \succ_i x_3 \succ_i x_4 \succ_i x_1 \); so by (IIA) we know that for every \( i=1,2,3,4 \):

\[
F(<x_i, x_{i+1}>) = x_i
\]

By applying the same way of reasoning as in Theorem 1 we can ensure that:

(a). \( x_j \notin <x_i> \forall i \neq j; \)

(b). \( x_j \notin <x_i, x_j> \forall j \neq 1,2,3,4, \; i \neq j-1; \)

(c). \( \forall B \subseteq A \) such that \( B \) has more than 2 alternatives or two non consecutive ones, \( F(<B>) \notin B \).

Moreover in this case we also know that

(d). \( F(<x_1, x_2>) \preceq w \) and \( F(<x_3, x_4>) \preceq w \) by applying (IM) since condition (c) together with (PO) implies that \( F(<x_1, x_2>) \) and \( F(<x_3, x_4>) \) are Interior Pareto Optimal points in \( <x_1, x_2> \) and \( <x_3, x_4> \) respectively.

By reasoning as in Theorem 1, if we consider the subset \( A = \{x_1, x_2, x_3, x_4\} \) and denote \( F(<A>) = w \), we can also ensure that \( w \notin A \) and that either \( w \in <x_2, x_4> \) or \( w \in <x_1, x_3> \). Without loss of generality we assume that \( w \in <x_1, x_3> \), \( x_1 \succ x_3 \) and we are going to analyze the possible location of alternatives of \( A \). The configuration of alternative \( x_1, x_2 \) and \( w \) as well as the different possibilities to locate alternative \( x_2 \) (which have been denoted as \( A_1, B_1, C_1, D_1 \) and \( E_1 \)) are shown in Fig. 11.

![Fig. 11](image)

**CASE 1:** \( x_2 \in A_1 \).

By applying (d), the possibilities to locate \( x_4 \) are \( C_1, D_1 \) or \( E_1 \). However in all of these situations by considering \( <x_1, x_2, x_4> \) and by applying (PO) together with (c), it is obtained that \( F(<x_1, x_2, x_4>) \) belongs to \( \text{seg}(x_1, x_4) \) or \( \text{seg}(x_1, x_4) \), both cases in contradiction with (IIA).
CASE 2: \( x_2 \in B_1 \).
By reasoning as above, possibilities for \( x_2 \) are given by \( C_2 \), \( D_2 \) or \( E_2 \) (see Fig.12). However if \( x_2 \) belongs to \( C_2 \) or \( D_2 \) it implies that \( x_2 \in \langle x_2, x_2 \rangle \), in contradiction with (b), and if \( x_2 \in E_2 \), then by considering \( \langle x_2, x_2, x_2 \rangle \) and reasoning as in Case 1, a contradiction is also obtained.

![Diagram](image1)

CASE 3: \( x_2 \in C_1 \).
In this case possibilities for \( x_2 \) are \( A_3 \), \( B_3 \), \( C_3 \) or \( D_3 \) (see Fig.13). But all of them imply a contradiction:

- if \( x_2 \in A_3 \) or \( D_3 \), the contradiction is obtained by considering \( \langle x_2, x_2, x_2 \rangle \) or \( \langle x_2, x_2, x_2 \rangle \), respectively, and by reasoning as in Case 1.
- if \( x_2 \in B_3 \) or \( C_3 \), then \( x_2 \in \langle x_2, x_2 \rangle \) or \( x_2 \in \langle x_2, x_2 \rangle \), respectively, both cases in contradiction with (b).

![Diagram](image2)

CASE 4: \( x_2 \in D_1 \).
Now Fig.14 shows possibilities for locating \( x_4 \) (\( A_4 \) and \( B_4 \)) but in both cases the same contradictions as in the previous case are obtained.

![Diagram](image3)

CASE 5: \( x_2 \in E_1 \).
Finally, in this case the way of reasoning is exactly the same as in Case 1 but by considering the subset \( \langle x_2, x_2, x_2 \rangle \).

Therefore we can conclude that there do not exist cycles of length 4.
Proof of Theorem 5: As in Theorem 1 we prove the acyclicity of the revealed preference by induction. By applying Theorem 6 and Lemma 1 we know the non existence of cycles of length 3 and 4, respectively. We assume that there are no cycles of length \( k \) \( \forall k < n \), \( n \geq 4 \) and we also want to prove the non existence of cycles of length \( n \).

By contradiction, assume that there are alternatives \( x_1, x_2, \ldots, x_n \in \mathbb{R}^2 \) such that \( x_1 \succ_P x_2 \succ_P \ldots \succ_P x_n \succ_P x_1 \). Therefore and by reasoning as in the previous cases it is obtained that:

1: \( x_j \not\in \langle x_i \rangle \) \( \forall i \neq j \)

2: \( x_j \not\in \langle x_i \rangle \) \( \forall i = 1, 2, \ldots, n, j \neq i+1 \)

3: \( \forall B \subseteq A \) such that \( B \) has two non consecutive alternatives or \( |B| \geq 3 \),

\[ F(B) \neq B \]

4: \( F(x_j, x_i) \leq w \) \( \forall j \in [1, n] \), \( \exists i \)

5: \( F(B) \leq w \) \( \forall B \subseteq A \): \( |B| \geq 3 \)

Moreover if we consider \( A = \{ x_1, x_2, \ldots, x_n \} \) and denote by \( w = F(A) \in \langle A \rangle \), we also know that \( w \in A \) and that there exist alternatives \( x_i, x_{i+k} \in A, k > 1 \), such that \( w \notin \langle x_i, x_{i+k} \rangle \) (in fact, by applying (c) and (PO), \( w \in \text{seg}(x_i, x_{i+k}) \)).

Without loss of generality we assume that \( x_i < x_{i+k} \). Now we analyze the configuration of alternatives \( x_{i+1}, x_{i+2}, \ldots, x_{i+k-1} \).

CASE 1. Assume that \( k \geq 2 \).
Possibilities for locating alternative \( x_{i+1} \) have been denoted as \( A_1, B_1, C_1, D_1 \) and \( E_1 \) (see Fig. 15)

![Diagram](image)

CASE 1.1. \( x_{i+1} \in A_1 \).
Then by applying (4) we know that \( x_{i+1} \) can be placed in \( C_1, D_1 \) or \( E_1 \), but in all of these cases we obtain a contradiction by considering \( F(x_{i+1}, x_i) \), since by (PO) and (3) we know that \( F(x_{i+1}, x_i) \) belongs to \( \langle x_{i+1}, x_i \rangle \) or \( \langle x_{i+1}, x_i \rangle \), both cases in contradiction with (IIA).

CASE 1.2. \( x_{i+1} \in D_1 \) or \( x_{i+1} \in E_1 \).
Alternative \( x_{i+1} \) can not be located in these areas since we are assuming that \( k > 2 \) and by (4) we know that \( F(x_{i+1}, x_i) \) \( \leq w \).

CASE 1.3. \( x_{i+1} \in B_1 \).
In this case we have to distinguish two possibilities:

1.3.a. \( k = 3 \).
In this case we analyze the location of $x_{i2}$, which can be placed in $B_2$, $C_2$, $D_2$, $E_2$ or $F_2$ (see Fig. 16).

Therefore the only possibility for locating alternative $x_{i2}$ is $C_2$ and moreover $x_{i2} \in <x_{i1}^{1}, x_{i3}^{1}>$, if not we have the same contradiction as in Case 1.1, by considering $<x_{i1}^{1}, x_{i2}^{1}, x_{i3}^{1}>$. So the configuration of these alternatives is as shown in Fig. 18.

However all of these possibilities imply a contradiction.

In concrete if $x_{i2} \in B_2$ or $x_{i2} \in F_2$, we obtain a contradiction by reasoning as in the Case 1.1 but by considering $<x_{i1}^{1}, x_{i2}^{1}>$ and $<x_{i1}^{1}, x_{i2}^{1}, x_{i3}^{1}>$ respectively.

If $x_{i2} \in E_2$, then the contradiction appears when we locate $x_{i4}$ ($x_{i4} \neq x_1$ since $n>4$), which has to be located in $C_2$ (from (4) $F(<x_{i1}^{1}, x_{i4}^{1}) \leq w$ and $F(<x_{i1}^{1}, x_{i4}^{1}) \leq w$), by considering $F(<x_{i3}^{1}, x_{i4}^{1}, x_{i4}^{1})$ and by reasoning as in Case 1.1.

If $x_{i2} \in D_2$, by reasoning as in the previous case, $x_{i4}$ has to be located in $C_2$ (see Fig. 17), but in this case $x_{i2} \in <x_{i1}^{1}, x_{i4}^{1}>$, in contradiction with (4).
Now, by applying (4) we know that $x_{i_4}$ can be located in $B_4$, $C_4$, $D_4$ or $E_4$.

$B_4$: if $x_{i_4} \in B_4$, then $x_{i_4} \in <x_{i_4}^2, x_{i_4}^3>$, in contradiction with (4).

$E_4$: if $x_{i_4} \in E_4$, then by considering $F(<x_{i_4}^1, x_{i_4}^2, x_{i_4}^3>)$ a contradiction as in Case 1.1 is obtained.

$C_4$, $D_4$: if $x_{i_4} \in C_4$ or $D_4$ and $x_{i_4} = x_i$ then we have a contradiction with (4) since in this case $x_{i_4} \in <x_{i_4}^1, x_{i_4}^3>$. If $x_{i_4} \neq x_i$ then the possibility for locating $x_{i_{n-1}}$ by reasoning in a way analogous to that of $x_{i_{n-1}}$ is below the horizontal line which passes through $x_{i_{n-2}}$ (if not, a contradiction with (2) is obtained); and the same reasoning can be applied to all of the alternatives until alternative $x_{i_{n-1}}$ is reached. In this moment a contradiction is obtained since this alternative has to be below the previously mentioned line, but it implies that $x_{i_{n-1}} \in <x_{i_1}, x_{i_{n-1}}>$, a contradiction with (2).

1.3b. \( k \neq 3 \).

In this case and by following a similar argument to the one used in the previous case we obtain that the final configuration of alternatives $x_i$, $x_{i_1}^1$, ..., $x_{i_{n-1}}^1$, $x_{i_k}$ is the one which is shown in Fig. 19. The contradiction which is obtained is the same as in the previous case when we try to locate alternatives $x_{i_{n-1}}$, ..., $x_{i_{n-1}}$.

CASE 1.4. $x_{i_{n-1}} \in C_1$.

The contradiction and the way of reasoning is exactly the same as in the case of $B_1$, although configuration of alternatives will be the one which is shown in Fig. 20.

CASE 2. $k = 2, n > 4$.

Possibilities for $x_{i_{n-1}}$ have been denoted as $A_2$, $B_2$, $C_2$, $D_2$ and $E_2$ (see Fig. 21). If $x_{i_{n-1}} \in A_2$, contradiction is obtained when we locate $x_{i_{n-1}}$ since by applying (4) it has to be placed in $C_2$, $D_2$ or $E_2$, but then by
considering \(<x_1, x_2, x_3>\) we obtain the same contradiction as in Case 1.1.

If \(x_{i+1} \in E_5\) then \(x_{i+3}\) has to be placed in \(A_5, B_5, C_5\) or \(D_5\) (see Fig. 21), but in each of these situations a contradiction is obtained by considering \(<x_1, x_2, x_3>\) and by reasoning again as in Case 1.1.

![Diagram 21](image-url)

With respect to \(B_5, C_5\) and \(D_5\), the kind of reasoning which implies the contradiction is analogous to the previous cases:

if \(x_{i+1} \in B_5\) then, by applying (4), \(x_{i+1}\) has to be located in \(C_5\) (see Fig. 22), which is a contradiction with (2) \((x_{i+1} \in <x_{i+2}, x_{i+3}>\).

![Diagram 22](image-url)

if \(x_{i+1} \in C_5\) the way of reasoning in order to obtain the contradiction is exactly the same as in the case of \(C_5\) or \(D_5_5\).

if \(x_{i+1} \in D_5\), then the contradiction is obtained when we locate \(x_{i+3}\), since by (4) it has to be placed in \(C_5\) (see Fig. 23) in contradiction with (2) \((x_{i+1} \in <x_{i+2}, x_{i+3}>\).

![Diagram 23](image-url)

So under the assumption of the existence of a cycle of length \(n\) we have proved that any of the possibilities for locating the alternatives of the cycle imply a contradiction, therefore we can conclude that there are no cycles of length \(n\).
REFERENCES


PUBLISHED ISSUES


WP-AD 93-10 *Dual Approaches to Efficiency* M. Browning. October 1993.


*Please contact IVIE’s Publications Department to obtain a list of publications previous to 1993.*
WP-AD 94-05  "Fair Allocation in a General Model with Indivisible Goods"  

WP-AD 94-05  "Honesty Versus Progressiveness in Income Tax Enforcement Problems"  

WP-AD 94-07  "Existence and Efficiency of Equilibrium in Economies with Increasing Returns to Scale: An Exposition"  

WP-AD 94-08  "Stability of Mixed Equilibria in Intersections Between Two Populations"  

WP-AD 94-09  "Imperfectly Competitive Markets, Trade Unions and Inflation: Do Imperfectly Competitive Markets Transmit More Inflation Than Perfectly Competitive Ones? A Theoretical Appraisal"  

WP-AD 94-10  "On the Competitive Effects of Divisionalization"  

WP-AD 94-11  "Efficient Solutions for Bargaining Problems with Claims"  

WP-AD 94-12  "Existence and Optimality of Social Equilibrium with Many Convex and Nonconvex Firms"  

WP-AD 94-13  "Revealed Preference Axioms for Rational Choice on Nonfinite Sets"  

WP-AD 94-14  "Market Learning and Price Dispersion"  

WP-AD 94-15  "Bargaining with Reference Points - Bargaining with Claims: Egalitarian Solutions Reexamined"  

WP-AD 94-16  "The Importance of Fixed Costs in the Design of Trade Policies: An Exercise in the Theory of Second Best"  

WP-AD 94-17  "Computers, Productivity and Market Structure"  

WP-AD 94-18  "Fiscal Policy Restrictions in a Monetory System: The Case of Spain"  

WP-AD 94-19  "Pareto Optimal Improvements for Sunsets: The Golden Rule as a Target for Stabilization"  

WP-AD 95-01  "Cost Monotonic Mechanisms"  

WP-AD 95-02  "Implementation of the Walrasian Correspondence by Market Games"  

WP-AD 95-03  "Terms of Trade and the Current Account: A Two-Country/Two-Sector Growth Model"  

WP-AD 95-04  "Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets"  

WP-AD 95-05  "Implementation of Stable Solutions to Marriage Problems"  

WP-AD 95-06  "Capabilities and Utilities"  

WP-AD 95-07  "Rational Choice on Nonfinite Sets by Means of Expansion-Contraction Axioms"  

WP-AD 95-08  "Veto in Fixed Agenda Social Choice Correspondences"  

WP-AD 95-09  "Temporary Equilibrium Dynamics with Bayesian Learning"  

WP-AD 95-10  "Existence of Maximal Elements in a Binary Relation Relaxing the Convexity Condition"  

WP-AD 95-11  "Three Kinds of Utility Functions from the Measure Concept"  

WP-AD 95-12  "Classical Equilibrium with Increasing Returns"  

WP-AD 95-13  "Bargaining with Claims in Economic Environments"  

WP-AD 95-14  "The Theory of Implementation when the Planner is a Player"  

WP-AD 95-15  "Popular Support for Progressive Taxation"  

WP-AD 95-16  "Expanded Version of Regret Theory: Experimental Test"  

WP-AD 95-17  "Unified Treatment of the Problem of Existence of Maximal Elements in Binary Relations. A Characterization"  

WP-AD 95-18  "A Note on Stability of Best Reply and Gradient Systems with Applications to Imperfectly Competitive Models"  

WP-AD 95-19  "Redistribution and Individual Characteristics"  

WP-AD 95-20  "A Mechanism for Meta-Bargaining Problems"  

WP-AD 95-21  "Signalling Games and Incentive Dominance"  
WP-AD 95-22  "Multiple Adverse Selection"

WP-AD 95-23  "Ranking Social Decisions without Individual Preferences on the Basis of Opportunities"

WP-AD 95-24  "The Extended Claim-Egalitarian Solution across Cardinalities"

WP-AD 95-25  "A Decent Proposal"

WP-AD 96-01  "A Spatial Model of Political Competition and Proportional Representation"
I. Ortuño. February 1996.

WP-AD 96-02  "Temporary Equilibrium with Learning: The Stability of Random Walk Beliefs"
S. Chatterji. February 1996.

WP-AD 96-03  "Marketing Cooperation for Differentiated Products"
M. Puig. February 1996.

WP-AD 96-04  "Individual Rights and Collective Responsibility: The Rights-Egalitarian Solution"

WP-AD 96-05  "The Evolution of Walrasian Behavior"

WP-AD 96-06  "Evolving Aspirations and Cooperation"

WP-AD 96-07  "A Model of Multiproduc Price Competition"

WP-AD 96-08  "Numerical Representation for Lower Quasi-Continuos Preferences"

WP-AD 96-09  "Rationality of Bargaining Solutions"